

# Fundamentals of Communications

## Engineering

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**Class:** Second Year

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**Room:** Comm-02

**Lecture: 10**

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## Energy & Power Spectral Densities °°

### \* The Energy Spectral density

according to Parseval's theorem, the energy in time domain can be related to Frequency domain as follows:-

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \underbrace{\int_{-\infty}^{\infty} |F(f)|^2 df}_{\text{Total energy}}$$

∴  $|F(f)|^2$  is the energy spectral density (ESD) of the signal  $f(t)$ .

\* The ESD is defined as

$$G(f) \triangleq |F(f)|^2 \quad \text{J/Hz}$$

$$\boxed{E = \int_{-\infty}^{\infty} G(f) df}$$

## \* Power Spectral Density (PSD)

We can find for a power signal the power spectral density (PSD) as follows

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

⇒ if  $x(t)$  is periodic, then

$$P = \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Hence ∴

$$P = \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \underbrace{S(f)}_{\text{PSD}} df$$

∴ The PSD is  $S(f)$ .

# Autocorrelation of an Energy Signal

- \* as we knew, correlation is a measure of similarity function, or we call it a matching process between two signals.
- \* If we do correlation between two different signals, then the process is called cross-correlation.
- \* If we do correlation between a signal & itself, then the process is called auto-correlation.
- \* However, the autocorrelation refers to the matching of a signal with a delayed version of itself.
- \* For a real-valued energy signal  $x(t)$ , the auto-correlation function can be defined as,

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau) dt \quad -\infty < \tau < \infty$$

NOTE  $R_x(\tau)$  is not a function of time  $(t)$ , it is a function of time-difference  $(\tau)$  between the waveform and its shifted copy.



⊕ The auto correlation function of a real-valued energy signal has the following properties:

① Symmetrical in  $\tau$  about zero,

$$R_x(\tau) = R_x(-\tau)$$

② Maximum value occurs at the origin,

$$R_x(\tau) \leq R_x(0) \text{ for all } \tau$$

③ The ESD and autocorrelation form a Fourier transform pair,

$$R_x(\tau) \xleftrightarrow{\text{F.T.}} \Psi_x(f)$$

④ The value at the origin is equal to the energy of the signal

$$R_x(0) = \int_{-\infty}^{\infty} x^2(t) dt$$

## \* Autocorrelation of a Power Signal

autocorrelation of a periodic signal (power signal).

\* For a real-valued power signal  $x(t)$ , the autocorrelation function can be defined as

$$R_x(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) x(t+\tau) dt \quad -\infty < \tau < \infty$$

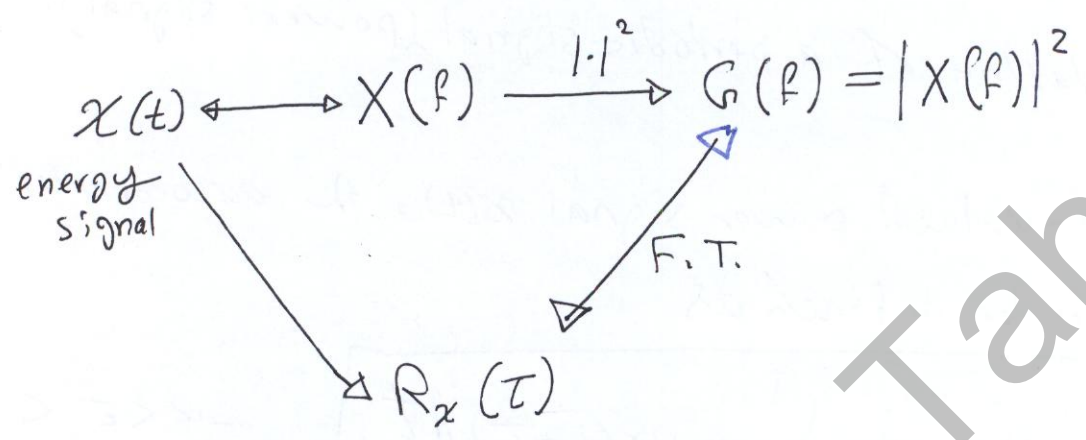
\* When  $x(t)$  power periodic signal, with period  $T_0$ , the time average may be taken over **single period  $T_0$** , and the autocorrelation function can be expressed as

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) x(t+\tau) dt \quad -\infty < \tau < \infty$$

\* The autocorrelation function of a real-valued periodic signal has properties similar to those of an energy signal: =

- ①  $R_x(\tau) = R_x(-\tau)$
- ②  $R_x(\tau) \leq R_x(0)$  for all  $\tau$
- ③  $R_x(\tau) \xleftrightarrow{F, T_0} S_x(f)$
- ④  $R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t) dt$

Keep this ESD diagram in your mind %



thus %  $R_x(\tau) = x(t) * x(-t)$   
 $= \int_{-\infty}^{\infty} x(t) x(t + \tau) dt$

and Do not forget

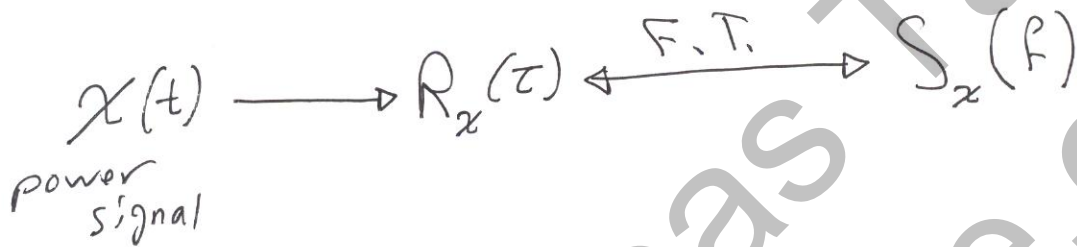
$$G(f) = \text{F.T.} \{R_x(\tau)\}$$

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Keep this PSD diagram in your mind ☺

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$



EX. consider the signal  $x(t) = A \cos(2\pi f_0 t + \theta)$ , find the autocorrelation for this  $x(t)$  signal and the Fourier transform of the autocorrelation function.

Solution

$$R_x(\tau) = \frac{1}{T_0} \int_0^{T_0} A^2 \cos(2\pi f_0 t + \theta) \cos(2\pi f_0 (t+\tau) + \theta) dt$$

$$= \frac{A^2}{2T_0} \int_0^{T_0} [\cos(2\pi f_0 \tau) + \cos(2\pi(2f_0)t + 2\pi f_0 \tau + 2\theta)] dt$$

$$= \frac{A^2}{2} \cos(2\pi f_0 \tau)$$

$$F.T. \{R_x(\tau)\} = S_x(f) = \frac{A^2}{4} [\delta(f - f_0) + \delta(f + f_0)]$$



# Common Notes about Autocorrelation

\* Assume  $R_x(\tau)$  is autocorrelation of  $x(t)$

if  $y(t) = A + x(t)$   
                  constant

$$R_y(\tau) = \int_{-\infty}^{\infty} y(t) y(t+\tau) dt = \int_{-\infty}^{\infty} [A + x(t)] [A + x(t+\tau)] dt$$

$$= \langle A^2 \rangle + \langle Ax(t+\tau) \rangle + \langle Ax(t) \rangle + \langle x(t)x(t+\tau) \rangle$$

$$= \underbrace{A^2 + 2A\langle x(t) \rangle}_{\text{constant}} + R_x(\tau)$$

\* Let  $z(t) = x(t - t_0)$

$$R_z(\tau) = \langle z(t) z(t+\tau) \rangle = \langle x(t-t_0) x(t-t_0+\tau) \rangle$$

$$= \langle x(\lambda) x(\lambda+\tau) \rangle$$

$$= R_x(\tau)$$

where  $\lambda = t - t_0$

\* ~~thus~~ autocorrelation is blind to time offsets

\* The autocorrelation function of a signal consists of two sinusoids is:

$$y(t) = x_1(t) + x_2(t)$$

$$\text{where } \left. \begin{aligned} x_1(t) &= A_1 \cos(2\pi f_1 t + \theta_1) \\ x_2(t) &= A_2 \cos(2\pi f_2 t + \theta_2) \end{aligned} \right\} f_1 \neq f_2$$

$$\therefore R_y(\tau) = R_{x_1}(\tau) + R_{x_2}(\tau)$$

Homework ∴

$$\text{prove } R_y(\tau) = R_{x_1}(\tau) + R_{x_2}(\tau)$$

$$\text{if } x_1(t) = A_1 \cos(2\pi f_1 t + \theta_1)$$

$$\text{and } x_2(t) = A_2 \cos(2\pi f_2 t + \theta_2)$$

$$\text{where } f_1 \neq f_2$$

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