Fundamentals of Communications Engineering

Department of Communications Engineering, College of Engineering, University of Diyala, 2016-2017

Class: Second Year
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Lecture: 10

Energy & Power Spectral Densities :

* The Energy Spectral density

according to Parseval's theorem, the energy in time domain can be related to Frequency domain as follows :-

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \underbrace{\int_{-\infty}^{\infty} |F(f)|^2 \, df}_{\text{Total energy}}$$

$\therefore$ $|F(f)|^2$ is the energy spectral density (ESD) of the signal $f(t)$.

* The ESD is defined as

$$G(f) \triangleq |F(f)|^2 \quad J/Hz$$

$$\therefore \boxed{E = \int_{-\infty}^{\infty} G(f) \, df}$$

* Power Spectral Density (PSD)

we can find for a power signal the power spectral density (PSD) as follows

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

⇒ if $x(t)$ is periodic, then

$$P = \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Hence :

$$P = \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} \underbrace{S(f)}_{PSD} \, df$$

∴ The PSD is $S(f)$.

Autocorrelation of an Energy Signal

* as we know, correlation is a measure of similarity function, or we call it a matching process between two signals.

* If we do correlation between two different signals, then the process is called cross-correlation.

* If we do correlation between a signal & itself, then the process is called auto-correlation.

* However, the autocorrelation refers to the matching of a signal with a delayed version of itself.

* For a real-valued energy signal $x(t)$, the auto-correlation function can be defined as,

$$\boxed{R_x(\tau) = \int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt} \qquad -\infty < \tau < \infty$$

NOTE: $R_x(\tau)$ is not a function of time (t), it is a function of time-difference (τ) between the waveform and its shifted copy.

the autocorrelation function of a real-valued energy signal has the following properties:-

1. Symmetrical in τ about zero,

$$R_x(\tau) = R_x(-\tau)$$

2. Maximum value occures at the origin,

$$R_x(\tau) \leq R_x(0) \quad \text{for all } \tau$$

3. The ESD and autocorrelation form a Fourier transform pair,

$$R_x(\tau) \xleftrightarrow{F.T.} \psi_x(f)$$

4. The value at the origin is equal to the energy of the signal

$$R_x(0) = \int_{-\infty}^{\infty} x^2(t)\, dt$$

* Autocorrelation of a Power Signal

autocorrelation of a periodic signal (power signal).

* For a real-valued power signal $x(t)$, the autocorrelation function can be defined as

$$\boxed{R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,x(t+\tau)\,dt} \qquad -\infty < \tau < \infty$$

* when $x(t)$ power periodic signal, with period T_0, the time average may be taken over single period T_0, and the autocorrelation function can be expressed as

$$\boxed{R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,x(t+\tau)\,dt} \qquad -\infty < \tau < \infty$$

* The autocorrelation function of a real-valued periodic signal has properties similar to those of an energy signal:

1. $R_x(\tau) = R_x(-\tau)$
2. $R_x(\tau) \leq R_x(0)$ for all τ
3. $R_x(\tau) \xleftrightarrow{F.T.} S_x(f)$
4. $R_x(0) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^2(t)\,dt$

Keep this ESD diagram in your mind :

$$\begin{tikzcd} \underset{\text{energy signal}}{x(t)} \arrow[r, leftrightarrow] \arrow[dr] & X(f) \arrow[r, "|\cdot|^2"] & G(f) = |X(f)|^2 \\ & R_x(\tau) \arrow[ur, leftrightarrow, "F.T."'] & \end{tikzcd}$$

thus : $R_x(\tau) = x(t) * x(-t)$

$$= \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

and Do not forget

$$G(f) = F.T.\{R_x(\tau)\}$$

Keep this PSD diagram in your mind

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$\underset{\text{power signal}}{x(t)} \longrightarrow R_x(\tau) \xleftrightarrow{\text{F.T.}} S_x(f)$$

EX. consider the signal $x(t) = A\cos(2\pi f_0 t + \theta)$, find the autocorrelation for this $x(t)$ signal and the Fourier transform of the autocorrelation function.

solution

$$R_x(\tau) = \frac{1}{T_0}\int_0^{T_0} A^2 \cos(2\pi f_0 t + \theta)\cos(2\pi f_0(t+\tau) + \theta)\, dt$$

$$= \frac{A^2}{2T_0}\int_{T_0} \left[\cos(2\pi f_0 \tau) + \cos(2\pi(2f_0)t + 2\pi f_0 \tau + 2\theta\right] dt$$

$$= \frac{A^2}{2}\cos(2\pi f_0 \tau)$$

$$\text{F.T.}\{R_x(\tau)\} = S_x(f) = \frac{A^2}{4}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

Common Notes about Autocorrelation :

* Assume $R_x(\tau)$ is autocorrelation of $x(t)$

if $y(t) = \underbrace{A}_{\text{constant}} + x(t)$

$$\therefore R_y(\tau) = \int_{-\infty}^{\infty} y(t)\, y(t+\tau)\, dt = \int_{-\infty}^{\infty} [A + x(t)][A + x(t+\tau)]\, dt$$

$$= \langle A^2 \rangle + \langle A x(t+\tau) \rangle + \langle A x(t) \rangle + \langle x(t)\, x(t+\tau) \rangle$$

$$= \underbrace{A^2 + 2A\langle x(t) \rangle}_{\text{constant}} + R_x(\tau)$$

* Let $z(t) = x(t - t_0)$

$$R_z(\tau) = \langle z(t)\, z(t+\tau) \rangle = \langle x(t-t_0)\, x(t - t_0 + \tau) \rangle$$

$$= \langle x(\lambda)\, x(\lambda + \tau) \rangle \qquad \text{where } \lambda = t - t_0$$

$$= R_x(\tau)$$

Thus : autocorrelation is blind to time offsets

* The autocorrelation function of a signal consists of two sinusoids is :

$$y(t) = x_1(t) + x_2(t)$$

where $x_1(t) = A_1 \cos(2\pi f_1 t + \theta_1)$ & $\left.\right\}$ $f_1 \neq f_2$
$x_2(t) = A_2 \cos(2\pi f_2 t + \theta_2)$

$$\therefore R_y(\tau) = R_{x_1}(\tau) + R_{x_2}(\tau)$$

Homework :

prove $R_y(\tau) = R_{x_1}(\tau) + R_{x_2}(\tau)$

if $x_1(t) = A_1 \cos(2\pi f_1 t + \theta_1)$

and $x_2(t) = A_2 \cos(2\pi f_2 t + \theta_2)$

where $f_1 \neq f_2$