❖ Numerical Integration

Unlike the differentiation, numerical integration is stable and more accurate for a given interpolating polynomial. This fact is illustrated in figure.

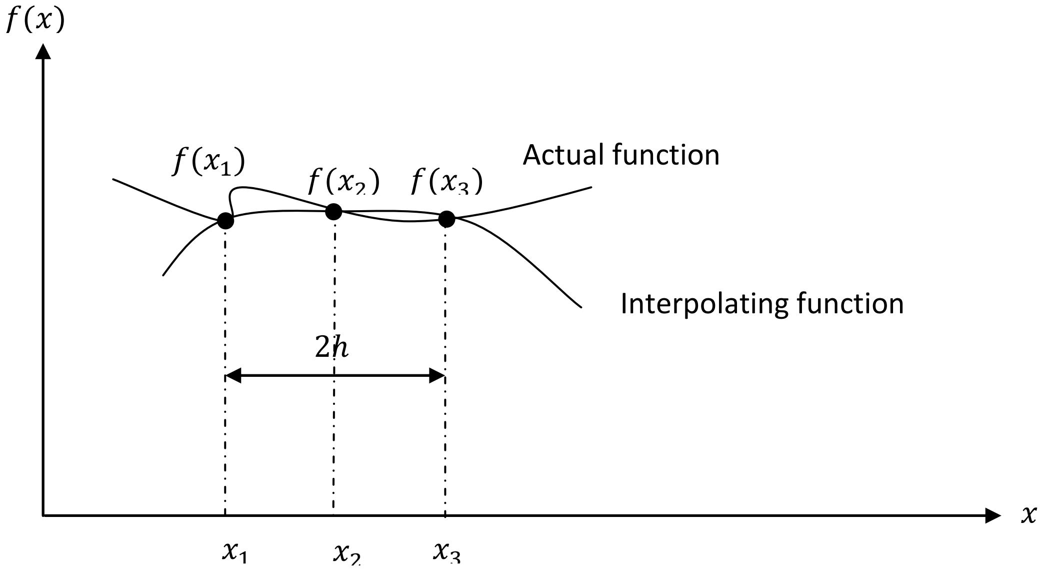

Fig (1)

1- Trapezoidal Rule of Integration

This method is one of the simplest integration methods available. It applies for equally spaced base points only. Consider the integral

$I = \int_{x_1}^{x_2} f(x)dx$

Consider figure (1), assuming that $F(x)$ is first order polynomial then

$F(x) = (a_0 + a_1 x) \rightarrow$

$I = \int_{x_1}^{x_2} (a_0 + a_1 x)\, dx$

$= \left[a_0 x + a_1 \frac{x^2}{2}\right]_{x_1}^{x_2}$

$= a_0 x_2 + a_1 \frac{{x_2}^2}{2} - a_0 x_1 - a_1 \frac{{x_1}^2}{2}$

$= a_0(x_2 - x_1) + \frac{a_1}{2}({x_2}^2 - {x_1}^2)$

The constant (a_0 *and* a_1) are determined by using the following conditions, if the exact value $F(x)$ and the numerical value is $f(x)$

At $x = x_1$: $F(x_1) = f(x_1)$

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This means that

$f(x_1) = a_0 + a_1 x_1$

$f(x_2) = a_0 + a_1 x_2$

Solving for (a_0 *and* a_1) to gate

$$a_0 = \frac{1}{x_2 - x_1}[x_2 f(x_1) - x_1 f(x_2)]$$

$$a_1 = \frac{1}{x_2 - x_1}[-f(x_1) + f(x_2)]$$

The result of integration will be

$I = \frac{1}{2}(x_2 - x_1)[f(x_1) + f(x_2)]$ since $(x_2 - x_1 = h)$ then

$I = \frac{1}{2} h\, [f(x_1) + f(x_2)]$ for more accuracy, more points between (x_1 *and* x_2) can be used therefore

$$I = \frac{1}{2} h\, [f(x_1) + 2f(x_1 + h) + 2f(x_1 + 2h) + \cdots f(x_2)]$$

Ex$_1$/ evaluate the following integral using trapezoidal rule and $h = 0.1$

$$I = \int_1^{1.6} e^{-x^2} dx$$

Sol:

$I = \frac{1}{2} h\, [f(1) + 2f(1.1) + 2f(1.2) + 2f(1.3) + 2f(1.4) + 2f(1.5) + f(1.6)]$

$= \frac{0.1}{2}(74.815547)$

= 3.740773

<u>Important note: </u>when the spaces between points are not equal then trapezoidal rule must be applied many times as will be illustrated in the following example.

Ex_2/ for the following data find the approximate area enclosed by them

x	1	2	3	5	8
y	1	2	4	5	6

Sol:

There are three (h)

$h_1 = 1$, $h_2 = 2$ and $h_3 = 3$

$I_1 = \frac{1}{2}h_1\,[f(1) + 2f(2) + f(3)] \rightarrow I_1 = \frac{1}{2}\,[1 + 2 * 2 + 4]$

$= \frac{9}{2}$

$I_2 = \frac{1}{2}h_2\,[f(3) + f(5)] \rightarrow I_2 = \;[4 + 5]$

$= 9$

$I_3 = \frac{1}{2}h_3\,[f(5) + f(8)] \rightarrow I_3 = \frac{1}{2} * 3 * \;[5 + 6]$

$= \frac{33}{2}$

$I = \frac{9}{2} + 9 + \frac{33}{2}$

= 29.5

2- Simpson's Rule of Integration Method

This method is more accurate than trapezoidal method, the disadvantage of this method is that this method can be applied to an even number of increment. The value of the integral for $x = -h$ to $x = h$ is approximated as follow:

$I = \int_{-h}^{h} f(x)dx = \int_{-h}^{h} F(x)dx$

$F(x) = a_0 + a_1x + a_2x^2 \rightarrow$

$I = \int_{-h}^{h}(a_0 + a_1x + a_2x^2\,)dx$

$= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3}\right]_{-h}^{h} \rightarrow \; I = 2ha_0 + 2\frac{h^3}{3}a_2$

Or $I = \left[2h \; 0 \; \frac{2h}{3}\right]\begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix}$

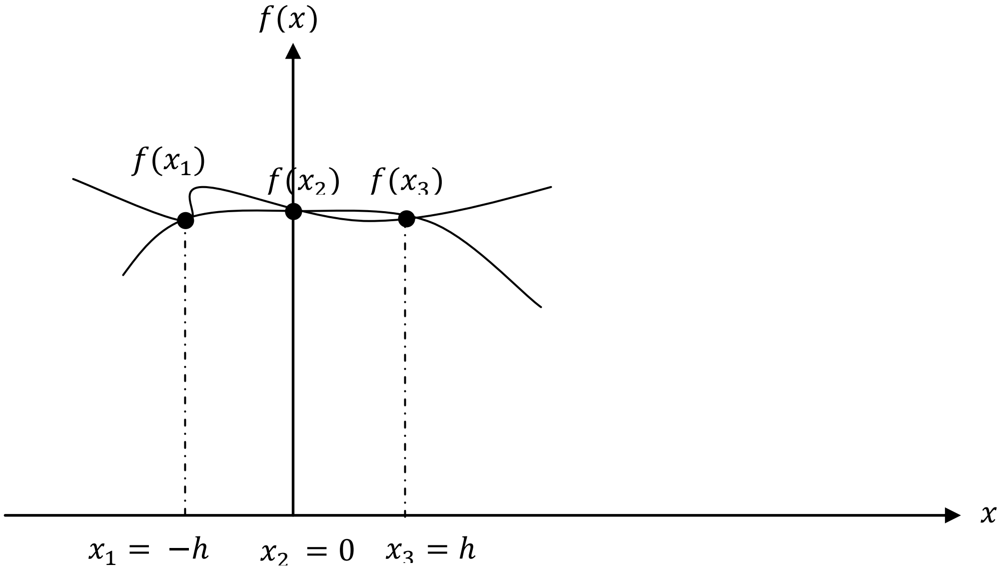

Fig (2)

From figure (2) the equation ($F(x) = a_0 + a_1 x + a_2 x^2$) can be written as

$F(x) = a_0 - a_1 h + a_2 h^2$

$F(x) = a_0 + a_1(0) + a_2(0)^2$

$F(x) = a_0 + a_1 h + a_2 h^2$

$\rightarrow$

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \rightarrow I = \left[2h \; 0 \; \frac{2h}{3}\right]\begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{h}{3} & \frac{4h}{3} & \frac{h}{3} \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

$$= \frac{h}{3}[f(x_1) + 4f(x_2) + f(x_3)]$$

The pervious equation may be expressed for (n) base points as follows:

$$I = \frac{h}{3}[f(x_1) + 4f(x_1 + h) + f(x_1 + 2h) + 4f(x_1 + 3h) + \cdots + f(x_n)]$$

Note that (n) must be an odd number.

Ex$_3$/ find the following integral using Simpson's Rule of Integration Method for $(h = 0.2)$

$$I = \int_1^{2.2} \sqrt[3]{x^2}dx$$

Sol:

since $I = \frac{h}{3}[f(x_1) + 4f(x_1 + h) + f(x_1 + 2h) + 4f(x_1 + 3h) + \cdots + f(x_n)]$ then

$$I = \frac{0.2}{3}[f(1) + 4f(1.2) + f(1.4) + 4f(1.6) + f(1.8) + 4f(2) + f(2.2)]$$

$$= \frac{0.2}{3}[1 + 4.517 + 1.2515 + 5.472 + 1.479 + 6.349 + 1.6915]$$

$= 1.451$

3- Development of Special Integration Formulas

In this type, integration formula are developed for cases in which the range of integration is bounded by fewer base points then the number of base points through which the interpolating function passes. For fourth-order interpolating function

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

Then $I = \int_{-h}^{h}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\text{dx}$

$$= \left[a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} + \frac{a_4 x^5}{5}\right]_{-h}^{h}$$

$$= 2h a_0 + \frac{2}{3} a_2 h^3 + \frac{2}{5}\, a_4 h^5$$

To find the constants $(a_0, a_1, a_2, a_3 \; and \; a_4)$ the following equation can be used

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

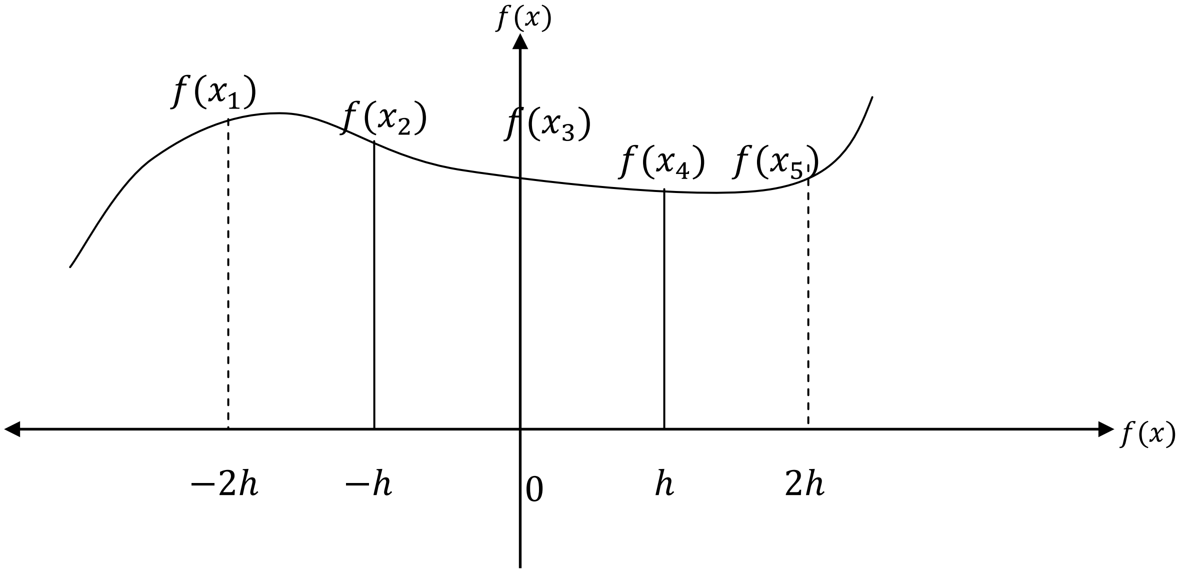

Fig (3)

From figure (3)

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \\ f(x_5) \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & -8 & 16 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \\ a_3 h^3 \\ a_4 h^4 \end{bmatrix}$$

Taking inverse to find that

$$a_0 = f(x_3)$$

$$a_2 = \frac{1}{24h^2}[-f(x_1) + 16f(x_2) - 30f(x_3) + 16f(x_4) - f(x_5)]$$

$$a_4 = \frac{1}{24h^4}[f(x_1) - 4f(x_2) + 6f(x_3) - 4f(x_4) + f(x_5)]$$

For the values of $(a_0, a_2, and \; a_4)$

$$I = \frac{h}{90}[-f(x_1) + 34f(x_2) + 114f(x_3) + 34f(x_4) - f(x_5)]$$

Ex4/ evaluate the following integral using Development of Special Integration Formulas, $h = 0.2$

$I = \int_1^{2.2} x^2\,dx$

Sol:

$f(1) = 1$, $f(1.3) = 1.69$, $f(1.6) = 2.56$, $f(1.9) = 3.61$

And $f(2.2) = 4.84$

Since

$$I = \frac{h}{90}[-f(x_1) + 34f(x_2) + 114f(x_3) + 34f(x_4) - f(x_5)] \rightarrow$$

$$I = \frac{0.2}{90}[-1 + 34 * (1.69) + 114 * 2.56 + 34 * (3.61) - 4.84]$$

= 1.036

4- Integration of Unevenly Spaced Base Points

when the points are not equally spaced as shown in figure (4) then it is evident that a second – order polynomial is needed for interpolating through three points:

$f(x) = a_0 + a_1x + a_2x^2$

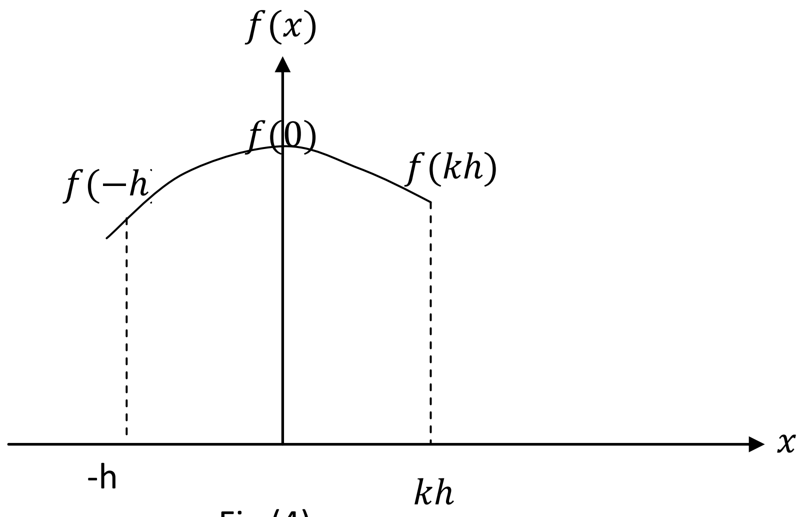

Fig (4)

From the equation of the second order and figure (4) it can be found that

$$\begin{bmatrix} f(-h) \\ f(0) \\ f(kh) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & k & k^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix}$$

$\rightarrow$

$$\begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix} = \frac{1}{k+k^2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & k & k^2 \end{bmatrix} \begin{bmatrix} f(-h) \\ f(0) \\ f(kh) \end{bmatrix}$$

$I = \int_{-h}^{kh} a_0 + a_1 x + a_2 x^2 \, dx$

$= \left[a_0 x + a_1 \frac{x^2}{2} + \frac{1}{3} a_2 x^3\right]_{-h}^{kh}$

$\rightarrow$

$I = \frac{h}{6(k+k^2)} [(-k^4 + 3k^2 + 2k) f(x_1) + (k+1)^4 f(x_2) + (2k^3 + 3k^2 - 1) f(x_3)]$

Ex$_5$/ find the integration of the following unevenly spaced points

$f(x)$	1	7	9
x	5	13	15

Sol:

$h = 13 - 5 = 8$

$hk = 15 - 13 = 2$ this means that $k = 0.25$

$I = \frac{8}{6(k+k^2)} [(-k^4 + 3k^2 + 2k) f(x_1) + (k+1)^4 f(x_2) + (2k^3 + 3k^2 - 1) f(x_3)]$

$\rightarrow \quad I = \frac{8}{6(0.25+0.0625)} [\left(-(\frac{1}{256}) + 3(0.0625) + 0.5\right) * 1 + \frac{625}{256} * 7 + \left(\frac{1}{32} + \frac{3}{16} - 1\right) * 9]$

5- Romberg's Integration

Given the following integral

$I = \int_A^B f(x)dx$

The trapezoidal rule gives

$I = \frac{h}{2}\,[f(A) + 2f(A+h) + \cdots + f(B)]\,,\ h = \frac{B-A}{(number\ of\ increment)}$

$\rightarrow I = I_0 + a_0h^2 + a_1h^4 + a_2h^6 + \cdots$

Romberg assume the following simple form

$I = I_0 + a_0h^2$

If $h = h_1\ gives\ I = I_1$ and if $h = h_2\ gives\ I = I_2$

$\rightarrow I = I_0 + a_0{h_1}^2\,, I = I_0 + a_0{h_2}^2$

This means that $I_0 = \frac{{h_2}^2 I_1 - {h_1}^2 I_2}{{h_2}^2 - {h_1}^2}$ or $I_0 = \frac{{h_{i+1}}^2 I_i - {h_i}^2 I_{i+1}}{{h_{i+1}}^2 - {h_i}^2}$ suppose that $h_i = 2h_{i+1} \rightarrow I_0 = \frac{1}{3}(4I_{i+1} - I_i)$

Ex$_6$/ evaluate the following integral using $(h = 4,2,1)$

$I = \int_0^4 e^x dx$

Sol: solutions

The exact solution

$I = \int_0^4 e^x dx$

$I = [e^x]_0^4$

$= 53.59815$

Taking $h_1 = 4$

$I_1 = \frac{h_1}{2}[f(0) + f(4)]$

$= \frac{4}{2}\,(e^0 + e^4)$

$= 111.1963$

For $h_2 = 2 \rightarrow I_2 = \frac{h_2}{2}[f(0) + 2f(2) + f(4)]$

$=\frac{2}{2}\ (e^0 + 2e^2 + e^4)$

$= 70.376262$

$\rightarrow I_{01} = \frac{1}{3}[4 * (70.376262) - 111.19]$

$=56.76958$

For $h_3 = 1\ \rightarrow$

$I_3 = \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)]$

$= 57.99195$

$I_{12} = \frac{1}{3}[4 * (57.99195) - 70.376262]$

$= 53.863846$

To more accuracy

$I = I_0 + a_0h^2 + a_1h^4$

If $h = h_1\ gives\ I = I_1, h = h_3\ gives\ I = I_2$ and if $h = h_2\ gives\ I = I_3$

$I = I_0 + a_0{h_1}^2 + a_1{h_1}^4$

$I = I_0 + a_0{h_2}^2 + a_1{h_2}^4$

$I = I_0 + a_0{h_3}^2 + a_1{h_3}^4$

Let $h_1 = 2h_2 = 4h_3$

$\rightarrow I_0 = \frac{1}{45}(I_1 - 20I_2 + 64I_3)$

EX_7/ repeat example (6) by using the equation $I_0 = \frac{1}{45}(I_1 - 20I_2 + 64I_3)$

Sol:

$I_1 = 111.1963$, $I_2 = 70.376262$ and $I_3 = 57.99195$ then

$\rightarrow I_0 = \frac{1}{45}(111.1963 - 20 * (70.376262) + 64 * (57.99195\ )$

$= 53.67013$

6- Gauss Quadrature Formulas

If $I = \int_A^B f(x)dx$ then this integral can be transformed to a simpler form by linearly relating $(x)$to a new variable (r)

$x = c_0 + c_1 r$

Where $(c_0$ & $c_1)$ are determined by assuming new limits of integration that is

For $(x = A\ let\ r = -1)$and $(x = B\ let\ r = 1)$ from which

$A = c_0 - c_1$and $B = c_0 + c_1$ solving for c_0&c_1 then

$c_0 = \frac{1}{2}(B + A)$ and $c_1 = \frac{1}{2}(B - A)$

This leads to

$x = \frac{1}{2}(B + A) + \frac{1}{2}(B - A)r \rightarrow dx = \frac{1}{2}(B - A)dr$

$I = \int_A^B f(x)dx$

$= \frac{1}{2}(B - A)\int_{-1}^{1} f\left[\frac{1}{2}(B + A) + \frac{1}{2}(B - A)r\right]dr$

Ex8/ solve the following integration using Gauss Quadrature method

$I = \int_0^1 (e^x + 1)dx$

Sol: since

$I = \frac{1}{2}(B - A)\int_{-1}^{1} f\left[\frac{1}{2}(B + A) + \frac{1}{2}(B - A)r\right]dr$

Then $I = \frac{1}{2}(1 - 0)\int_{-1}^{1} f\left[\frac{1}{2}(1 + 0) + \frac{1}{2}(1 - 0)r\right]dr$

$= \frac{1}{2}\int_{-1}^{1} f\left[\frac{1}{2} + \frac{1}{2}r\right]dr \rightarrow I = \frac{1}{2}\int_0^1 \left(e^{(\frac{1}{2}+\frac{1}{2}r)} + 1\right)dr$

$= \frac{1}{2}\left[e^{(\frac{1}{2}+\frac{1}{2}r)} + 1\right]_{-1}^{1}$

= e