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❖ Introduction

The ultimate aim of the field of numerical analysis is to provide convenient methods for obtaining useful solutions to mathematical problems and for extracting useful information from available solutions which are not expressed in tractable forms. Such problems may each be formulated, for example, in terms of an algebraic or transcendental equation, an ordinary or partial differential equation, or an integral equation, or in terms of a set of such equations.

❖ Numerical Solution of Linear Equations

A linear equation is one in which a variable only appears to the first power in every terms of a given equation, thus a system of (m) linear equations in (n) unknowns (x_j) ,($j = 1,2,3,\dots\dots,n$) can be represented as follow:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, 2, \dots., m$$

Then the equations will be expanded as follow:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

. . . .

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$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

There are many methods to solve the linear equations such as:

1- Cramer's Rule

The basis for this method may be explained by considering the following example of two nonhomogeneous equations:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Then the variables (x_1, x_2) can be found as follow

$x_1 = \frac{|A_1|}{|A|}$, $x_1 = \frac{|A_2|}{|A|}$, where $(A, A_1,\ A_2)$ are the matrices

$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & b_1 \\ a_{12} & b_{22} \end{bmatrix}$ respectivelly

Ex$_1$/ use the Cramer's rule to solve the following equations

$$2x_1 - 3x_2 = 5$$

$$x_1 + x_2 = 5$$

Sol:

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 5 & -3 \\ 5 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}, and\ B = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$|A| = [(2 * 1) - (1 * -3)] \rightarrow |A| = 5$$

$$|A_1| = [(5 * 1) - (5 * -3)] \rightarrow |A_1| = 20$$

$$|A_2| = [(2 * 5) - (1 * 5)] \rightarrow |A_2| = 5$$

$\rightarrow$

$$x_1 = \frac{|A_1|}{|A|} \rightarrow x_1 = \frac{20}{5}$$

$= 4$

$$x_1 = \frac{|A_1|}{|A|} \rightarrow x_1 = \frac{20}{5}$$

$= 1$

2- Gauss's

For a general form (3*3) matrix shown

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where the matrix A with elements aij is assumed to be nonsingular. Moreover, we assume that no a_{ij} or b_i is zero. (This is simply to maintain complete generality.) It is clear from this representation that if $a_{21} = a_{31} =$

a_{32} = 0, the solution to the whole system can be calculated immediately, starting with

$x_3 = \frac{b_3}{a_{33}}$

and working backward, in order, to x_1.

In other words all the elements of the lowest triangle of the matrix (A) must be equal to zeros.

Ex$_2$/ solve with Gauss elimination method

$2x_1 - 4x_2 + x_3 = 2 \qquad ....R_1$

$4x_1 + x_2 + 2x_3 = 6 \qquad ....R_2$

$x_1 + x_2 + 3x_3 = 0 \qquad ....R_3$

Sol:

$$\begin{bmatrix} 2 & -4 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$$

$$\left[\begin{matrix} 2 & -4 & 1 \\ 4 & 1 & 2 \\ 1 & 1 & 3 \end{matrix} \;\vdots\; \begin{matrix} 2 \\ 6 \\ 0 \end{matrix}\right] \text{divide eq. one on (2)} \rightarrow \left[\begin{matrix} 1 & -2 & 0.5 \\ 4 & 1 & 2 \\ 1 & 1 & 3 \end{matrix} \;\vdots\; \begin{matrix} 1 \\ 6 \\ 0 \end{matrix}\right]$$

$(R_2 - 4 * R_3)$ and put the result in $R_3 \rightarrow$

$$\left[\begin{matrix} 1 & -2 & 0.5 \\ 4 & 1 & 2 \\ 0 & -3 & -10 \end{matrix} \;\vdots\; \begin{matrix} 1 \\ 6 \\ 6 \end{matrix}\right] (R_2 - 4 * R_1) \text{ and put the result in } R_2 \rightarrow$$

$$\left[\begin{matrix} 1 & -2 & 0.5 \\ 0 & 9 & 0 \\ 0 & -3 & -10 \end{matrix} \;\vdots\; \begin{matrix} 1 \\ 2 \\ 6 \end{matrix}\right] \left(\frac{R_2}{3} + R_3\right) \text{ and put the result in } R_3 \rightarrow$$

$$\left[\begin{matrix} 1 & -2 & 0.5 \\ 0 & 9 & 0 \\ 0 & 0 & -10 \end{matrix} \;\vdots\; \begin{matrix} 1 \\ 2 \\ \frac{20}{3} \end{matrix}\right] \rightarrow -10x_3 = \frac{20}{3} \rightarrow x_3 = \frac{-2}{3}$$

$9x_2 = 2 \rightarrow \; x_2 = \frac{2}{9} \rightarrow \; x_1 - 2 * \frac{2}{9} + 0.5 * \frac{-2}{3} = 1 \rightarrow \; x_1 = 1 + (\frac{4}{9} + \frac{1}{3})$

$\therefore x_1 = \frac{16}{9}$

3- Gauss-Jordan Elimination Method

This method is similar to the Gauss one except that the elements of the upper and lower triangles of the matrix must be zeros, which means that

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

This method needs more effort than Gauss method but it is provides direct results.

Ex_2/ solve the following system

$2x_1 - 4x_2 + 6x_3 = 5 \qquad \ldots\ldots.R_1$

$x_1 + 3x_2 - 7x_3 = 2 \qquad \ldots\ldots.R_2$

$7x_1 + 5x_2 + 9x_3 = 4 \qquad \ldots\ldots.R_3$

Sol:

$$\begin{bmatrix}2 & -4 & 6\\1 & 3 & -7\\7 & 5 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}5\\2\\4\end{bmatrix}$$

$$\left[\begin{matrix}2 & -4 & 6\\1 & 3 & -7\\7 & 5 & 9\end{matrix}\ \vdots\ \begin{matrix}5\\2\\4\end{matrix}\right]\left(\frac{R_1}{2}\right) \rightarrow \left[\begin{matrix}1 & -2 & 3\\1 & 3 & -7\\7 & 5 & 9\end{matrix}\ \vdots\ \begin{matrix}2.5\\2\\4\end{matrix}\right](R_2 - R_1)$$ put the result in R_2 &$(R_3 - 7R_1)$ and put the result in R_3

$$\left[\begin{matrix}1 & -2 & 3\\0 & 5 & -10\\0 & 19 & -12\end{matrix}\ \vdots\ \begin{matrix}2.5\\-0.5\\-13.5\end{matrix}\right]\left(\frac{R_1}{2}\right) \rightarrow \left[\begin{matrix}1 & -2 & 3\\0 & 1 & -2\\0 & 19 & -12\end{matrix}\ \vdots\ \begin{matrix}2.5\\-0.1\\-13.5\end{matrix}\right]$$

$(2R_2 + R_1)$and put the result in R_1 &$(R_3 - 19R_2)$ and put the result in R_2

$$\left[\begin{matrix}1 & 0 & -1\\0 & 1 & -2\\0 & 0 & 26\end{matrix}\ \vdots\ \begin{matrix}2.3\\-0.1\\-11.6\end{matrix}\right]\left(\frac{R_3}{26}\right)\text{on } (26) \rightarrow \left[\begin{matrix}1 & 0 & -1\\0 & 1 & -2\\0 & 0 & 1\end{matrix}\ \vdots\ \begin{matrix}2.3\\-0.1\\-0.44\end{matrix}\right]$$

$(R_3 + R_1)$ and put the result in R_1&$(2R_3 + R_2)$ and put the result in R_2

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & 1.85 \\ 0 & 1 & 0 & -0.99 \\ 0 & 0 & 1 & -0.44 \end{array}\right]$$

This means that x_1 = 1.85, x_2 =−0.99,x_3 =−0.44

Ex3/ solve the following system

$x_1 - 2x_2 + 3x_3 = 1$ …….R_1

$2x_1 + x_2 + x_3 = 7$ …….R_2

$3x_1 + 2x_2 - 2x_3 = 5$ …….R_3

Sol:

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix} \rightarrow \left[\begin{array}{ccc:c} 1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 7 \\ 3 & 2 & -2 & 5 \end{array}\right]$$

$(2R_1 - R_2)$and put the result in R_2 &$(3R_1 - R_3)$ and put the result in R_3

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 1 \\ 0 & -5 & 5 & -5 \\ 0 & -8 & 11 & -2 \end{array}\right]\left(\frac{R_2}{(-5)}\right) \rightarrow \left[\begin{array}{ccc:c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -8 & 11 & -2 \end{array}\right](8R_2 + R_3)$$

and put the result in R_3 $\rightarrow \left[\begin{array}{ccc:c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 6 \end{array}\right]\left(\frac{R_3}{3}\right) \rightarrow$

$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $(R_2 + R_3)$ and put the result in R_2 $\rightarrow$

$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $(2R_2 + R_1)$and put the result in R_1 $\rightarrow$

$\left[\begin{array}{ccc:c} 1 & 0 & 3 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $(R_1 - 3R_3)$ and put the result in R_1 $\rightarrow$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This means that x_1 = 1, x_2 = 3,x_3 = 2

Ex$_4$/ solve the following equations using Gauss-Jordan method

$x_1 - 4x_2 + 3x_3 = A$ …….R_1

$3x_1 + 2x_2 - x_3 = B$ …….R_2

$-5x_1 + x_2 + 3x_3 = C$ …….R_3

Sol:

$$\begin{bmatrix} 1 & -4 & 3 \\ 3 & 2 & -1 \\ -5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ 5 \end{bmatrix} \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 3 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ -5 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$ $(3R_1 - R_2)$ and put the result in R_2, $(5R_1 + R_3)$ and put the result in R_3 $\rightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 3 & 1 & 0 & 0 \\ 0 & -14 & 10 & 3 & 1 & 0 \\ 0 & -19 & 18 & 5 & 0 & 1 \end{array}\right]$$ $\left(\frac{19}{14} R_2 - R_3\right)$ and put the result in R_3 $\rightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 3 & 1 & 0 & 0 \\ 0 & -14 & 10 & 3 & 1 & 0 \\ 0 & 0 & \frac{-31}{7} & \frac{-13}{14} & \frac{19}{14} & 1 \end{array}\right] \left(\frac{-7}{31}\right) R_3 \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 3 & 1 & 0 & 0 \\ 0 & -14 & 10 & 3 & 1 & 0 \\ 0 & 0 & 1 & \frac{13}{62} & \frac{-19}{62} & \frac{-7}{31} \end{array}\right]$$ $(10R_3 - R_2)$ and put the result in R_2,

$(3R_3 - R_1)$ and put the result in R_1 $\rightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 0 & \frac{-23}{62} & \frac{-57}{62} & \frac{-21}{31} \\ 0 & -14 & 0 & \frac{-28}{31} & \frac{-126}{31} & \frac{-70}{31} \\ 0 & 0 & 1 & \frac{13}{62} & \frac{-19}{62} & \frac{-7}{31} \end{array}\right] \left(\frac{R_2}{-14}\right) \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 0 & \frac{-23}{62} & \frac{-57}{62} & \frac{-21}{31} \\ 0 & 1 & 0 & \frac{2}{31} & \frac{9}{31} & \frac{5}{31} \\ 0 & 0 & 1 & \frac{13}{62} & \frac{-19}{62} & \frac{-7}{31} \end{array}\right]$$ $(4R_2 + R_1)$ and put the result in R_1 $\rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{-7}{62} & \frac{15}{62} & \frac{-1}{31} \\ 0 & 1 & 0 & \vdots & \frac{2}{31} & \frac{9}{31} & \frac{5}{31} \\ 0 & 0 & 1 & \vdots & \frac{13}{62} & \frac{-19}{62} & \frac{-7}{31} \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-7}{62} & \frac{15}{62} & \frac{-1}{31} \\ \frac{2}{31} & \frac{9}{31} & \frac{5}{31} \\ \frac{13}{62} & \frac{-19}{62} & \frac{-7}{31} \end{bmatrix} \begin{bmatrix} A \\ B \\ 5 \end{bmatrix}$$ this mean that

$x_1 = \frac{-7}{62}\text{A} + \frac{15}{62}\text{B} - \frac{1}{31}\text{C}$

$x_2 = \frac{2}{31}\text{A} + \frac{9}{31}\text{B} - \frac{5}{31}\text{C}$

$x_3 = \frac{13}{62}\text{A} - \frac{19}{62}\text{B} - \frac{7}{31}\text{C}$

4- Doolittle's Method

Before studying this method, it is important to learn how any square matrix can be expressed as an upper and lower triangular ($[A] = [L][U]$), so that

If $[A]\,[X] = [B]$ then $[L][U][X] = [B]$, assume that $[Y] = [U][X]$then

$[L][Y] = [B]$

How (u_{11}&L_{11}) can be chosen so that $u_{11}L_{11} = a_{11}$

$u_{12} = \frac{a_{12}}{L_{11}}$, $u_{13} = \frac{a_{13}}{L_{11}}$ & $L_{21} = \frac{a_{21}}{u_{11}}$, $L_{31} = \frac{a_{31}}{u_{11}}$

In general $L_{jk} = \frac{1}{u_{kk}}(a_{jk} - \sum_{s=1}^{k-1} L_{js}\, u_{sk})$ for $j = k+1, \dots, n,\ k \geq 2$

Ex$_5$/ decompose the matrix $[A]$ as$[L][U]$

$$A = \begin{bmatrix} 8 & -4 & 8 \\ 4 & 6 & 2 \\ 16 & 4 & 3 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Sol:

Let $u_{11} = 4, L_{11} = 2$ so that $u_{11}L_{11} = 8$

$L_{11}u_{12} \rightarrow u_{12} = \frac{a_{12}}{L_{11}} \rightarrow u_{12} = \frac{-4}{2} \rightarrow u_{12} = -2$

$L_{11}u_{13} \rightarrow u_{13} = \frac{a_{13}}{L_{11}} \rightarrow u_{13} = \frac{8}{2} \rightarrow u_{13} = 4$

$L_{21}u_{11} \rightarrow L_{21} = \frac{a_{21}}{u_{11}} \rightarrow L_{21} = \frac{4}{4} \rightarrow L_{21} = 1,\ L_{31}u_{11} \rightarrow L_{31} = \frac{a_{31}}{u_{11}} \rightarrow L_{31} = \frac{16}{4} \rightarrow L_{31} = 4$

$u_{22} = \frac{a_{22} - L_{21}u_{12}}{L_{22}} \rightarrow u_{22}L_{22} = 8 \rightarrow u_{22} = 2\ \&\ L_{22} = 4$

$L_{31}u_{12} + L_{32}u_{22} = a_{32} \rightarrow L_{32} = \frac{4-(4*-2)}{2} \rightarrow L_{32} = 6$

$L_{21}u_{13} + L_{22}u_{23} = a_{23}$

$u_{23} = \frac{2-4}{4} \rightarrow u_{23} = \frac{-1}{2}$

$u_{33}L_{33} = a_{33} - L_{31}u_{13} - L_{32}u_{23} \rightarrow u_{33}L_{33} = 3 - 16 + 3$

$\rightarrow L_{33}u_{33} = -10 \rightarrow L_{33} = 5\ \&\ u_{33} = -2$

This means that $\begin{bmatrix} 8 & -4 & 8 \\ 4 & 6 & 2 \\ 16 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 4 & 6 & 5 \end{bmatrix} \begin{bmatrix} 4 & -2 & 4 \\ 0 & 2 & -0.5 \\ 0 & 0 & -2 \end{bmatrix}$

The Doolittle's method states that $(L_{ii} = 0)$ f0r $(i = 1, 2, 3, \dots)$

Or

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

In this case

$u_{11} = a_{11}, u_{12} = a_{12}, u_{13} = a_{13}$

$L_{21} = \frac{a_{21}}{u_{11}}, L_{31} = \frac{a_{31}}{u_{11}},$

$u_{22} = a_{22} - L_{21}a_{12}, u_{23} = a_{23} - L_{21}u_{13}$

$u_{33} = a_{33} - (L_{31}u_{13}) - (L_{32}u_{23})$

$L_{32} = \frac{a_{32} - L_{31}u_{12}}{u_{22}} \rightarrow L_{32} = \frac{a_{32} - L_{31}u_{12}}{a_{22} - L_{21}a_{12}}$

Ex$_6$/ solve the following equations using Doolittle's method

$2x_1 + x_2 + 2x_3 = 1 \qquad ....R_1$

$4x_1 + 6x_2 + 9x_3 = 1 \qquad ....R_2$

$12x_2 + 18x_3 = -6 \qquad ....R_3$

Sol:

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 6 & 9 \\ 0 & 12 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = a_{11} \rightarrow u_{11} = 2, u_{12} = a_{12} \rightarrow u_{12} = 1, u_{13} = a_{13} \rightarrow u_{13} = 2$

$L_{21} = \frac{a_{21}}{a_{11}} \rightarrow L_{21} = \frac{4}{2} \rightarrow L_{21} = 2$

$L_{31} = \frac{a_{31}}{a_{11}} \rightarrow L_{31} = \frac{0}{2} \rightarrow L_{31} = 0$

$u_{22} = a_{22} - L_{21}a_{12} \rightarrow u_{22} = 6 - (2 * 1) \rightarrow u_{22} = 4$

$L_{32} = \frac{a_{32} - L_{31}u_{12}}{u_{22}} \rightarrow L_{32} = \frac{12-(0*1)}{4} \rightarrow L_{32} = 3$

$u_{23} = 9 - (2 * 2) \rightarrow u_{23} = 5$

$u_{33} = a_{33} - (L_{31}u_{13}) - (L_{32}u_{23}) \rightarrow u_{33} = 18 - (0 * 2) - (3*5)$

$=3$

$$\therefore \begin{bmatrix} 2 & 1 & 2 \\ 4 & 6 & 9 \\ 0 & 12 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Since $[A]\,[X] = [B] \rightarrow [L][U][X] = [L][Y] = [B]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} [Y] = [B] \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix}$$

$\therefore y_1 = 1, 2y_1 + y_2 = 1 \rightarrow y_2 = -1 \,\&\, 3y_2 + y_3 = -6 \rightarrow y_3 = -3$

Then $[U][X] = [Y] \rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

$\therefore 3x_3 = -3 \rightarrow x_3 = -1, \; 4y_2 + 5\,y_3 = -1 \rightarrow x_2 = 1\&$

$2x_1 + x_2 + 2x_3 = 1 \rightarrow x_1 = 1 \rightarrow [X] = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

5- Crout's Method

It is the same as Doolittle's method except that $[L]$ is required to have main diagonal of unity.

Ex$_7$/ solve the equations using Doolittle's method

$2x_1 + 3x_2 + x_3 = 1 \qquad ....R_1$

$-2x_1 + 2x_2 + 4x_3 = 1 \qquad ....R_2$

$x_1 + 3x_2 + 8x_3 = -6 \qquad ....R_3$

Sol:

First using Doolittle's method

$$\begin{bmatrix} 2 & 3 & 1 \\ -2 & 2 & 4 \\ 1 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = a_{11} \rightarrow u_{11} = 2\,, u_{12} = a_{12} \rightarrow u_{12} = 3, u_{13} = a_{13} \rightarrow u_{13} = 1$

$L_{21} = \frac{a_{21}}{u_{11}} \rightarrow L_{21} = \frac{-2}{2} \rightarrow L_{21} = -1$

$L_{31} = \frac{a_{31}}{u_{11}} \rightarrow L_{31} = \frac{1}{2} \rightarrow L_{31} = \frac{1}{2}$

$u_{22} = a_{22} - L_{21}u_{12} \rightarrow u_{22} = 2 - (-1 * 3) \rightarrow u_{22} = 5$

$L_{32} = \frac{a_{32} - L_{31}u_{12}}{u_{22}} \rightarrow L_{32} = \frac{3-(0.5*3)}{5} \rightarrow L_{32} = \frac{3}{10}$

$u_{23} = a_{23} - (L_{21}u_{13}) \rightarrow u_{23} = 4 - (-1 * 1) \rightarrow u_{23} = 5$

$u_{33} = a_{33} - (L_{31}u_{13}) - (L_{32}u_{23}) \rightarrow u_{33} = 8 - (0.5 * 1) - (\frac{3}{10}*5)$

$=6$

$$\therefore \begin{bmatrix} 2 & 3 & 1 \\ -2 & 2 & 4 \\ 1 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0.5 & \frac{3}{10} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Since $[A]\,[X] = [B] \rightarrow [L][U][X] = [L][Y] = [B]$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0.5 & \frac{3}{10} & 1 \end{bmatrix}[Y] = [B] \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0.5 & \frac{3}{10} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix}$$

$\therefore y_1 = 1,\ -y_1 + y_2 = 1 \rightarrow y_2 = 2 \& \ 0.5y_1 + \frac{3}{10} y_2 + y_3 = -6 \rightarrow y_3 = -7.1$

Then $[U][X] = [Y] \rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -7.1 \end{bmatrix}$

$\therefore 6x_3 = -7.1 \rightarrow x_3 = \frac{-7.1}{6},\ 5x_2 + 5\,x_3 = 2 \rightarrow x_2 = \frac{19}{12} \&$

$$2x_1 + 3x_2 + x_3 = 1 \rightarrow x_1 = \frac{77}{60} \rightarrow [X] = \begin{bmatrix} \frac{77}{60} \\ \frac{19}{12} \\ \frac{-7.1}{6} \end{bmatrix}$$

Ex$_8$/ use Crout's method to solve following equations

$2x_1 + x_2 + x_3 = 4 \qquad ....R_1$

$x_1 + 2x_2 + x_3 = 2 \qquad ....R_2$

$x_1 + x_2 + 2x_3 = -9 \qquad ....R_3$

Sol:

First using Doolittle's method

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$L_{11} = a_{11} \rightarrow L_{11} = 2\ ,\ L_{21} = a_{21} \rightarrow L_{21} = 1,\ L_{31} = a_{31} \rightarrow L_{31} = 1$

$u_{11} = u_{22} = u_{33} = 1,\ u_{12} = \frac{a_{12}}{L_{11}} \rightarrow u_{12} = \frac{1}{2},\ u_{13} = \frac{a_{13}}{L_{11}} \rightarrow u_{13} = \frac{1}{2}$

$L_{21} = \frac{a_{21}}{u_{11}} \rightarrow L_{21} = \frac{1}{1} \rightarrow L_{21} = 1$

$L_{31} = \frac{a_{31}}{u_{11}} \rightarrow L_{31} = \frac{1}{1} \rightarrow L_{31} = 1$

$L_{32} = \frac{a_{32} - L_{31}u_{12}}{u_{22}} \rightarrow L_{32} = \frac{1-(1*0.5)}{1} \rightarrow L_{32} = \frac{1}{2}$

$$L_{22} = \frac{a_{22}-L_{21}u_{12}}{u_{11}} \rightarrow L_{22} = \frac{2-(1*0.5)}{1} \rightarrow L_{22} = \frac{3}{2}$$

$$u_{23} = \frac{a_{23}-L_{31}u_{12}}{L_{22}} \rightarrow u_{23} = \frac{1-(1*0.5)}{(\frac{3}{2})} \rightarrow u_{23} = \frac{1}{3}$$

$$L_{33} = \frac{a_{33}-L_{31}u_{12}-L_{32}u_{23}}{u_{33}} \rightarrow L_{33} = \frac{2-(1*0.5)-(0.5*\frac{1}{3})}{1} \rightarrow L_{33} = \frac{4}{3}$$

$$\therefore \begin{bmatrix} 2 & 3 & 1 \\ -2 & 2 & 4 \\ 1 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{3}{2} & 0 \\ 1 & \frac{1}{2} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

Since $[A]\,[X] = [B] \rightarrow [L][U][X] = [L][Y] = [B]$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} [Y] = [B] \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -8 \end{bmatrix}$$

$\therefore y_3 = -9 \,, \rightarrow\ y_2 + \left(\frac{1}{3} * -9\right) = 2 \rightarrow\ y_2 = 5$

In addition $y_1 + \frac{1}{2}\, y_2 + \frac{1}{2} y_3 = -8 \rightarrow y_1 + \left(\frac{1}{2} * 5\right) + \left(\frac{1}{2} * -9\right) = -9$

$y_1 = -7.5$

Then $[U][X] = [Y] \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{3}{2} & 0 \\ 1 & \frac{1}{2} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7.5 \\ 5 \\ -9 \end{bmatrix}$

$\therefore 2x_1 = -7.5 \rightarrow x_1 = \frac{-7.5}{2},\ x_1 + \frac{3}{2}\, x_2 = 5 \rightarrow\ x_2 = \frac{35}{6}$ &

$$x_1 + \frac{1}{2}x_2 + \frac{4}{3}x_3 = -9 \rightarrow\ x_1 = \frac{-49}{8} \rightarrow [X] = \begin{bmatrix} \frac{-7.5}{2} \\ \frac{35}{6} \\ \frac{-49}{8} \end{bmatrix}$$