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❖ Numerical Differentiation

For forward interpolation

$x = x_0 - rh \rightarrow r = \frac{x-x_0}{h}$, from it $x - x_1 = (r-1)h$

$f_n = f_0 + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + \cdots + (x-x_0) \ldots (x-x_n)f[x_0,x_1,\ldots.x_n]$

Now for $\Delta f_i = f_{i+1} - f_i \rightarrow \Delta^2 = \Delta f_{i+1} - \Delta f_i$ and so on

$\Delta^n = \Delta^{n-1} f_{i+1} - \Delta^{n-1} f_i$

Since $f[x_0,x_1] = \frac{f_1-f_0}{x_1-x_0}$, $\Delta f_0 = f_1 - f_0$ and $(x-x_0) = rh \rightarrow$

$$f_n = f_0 + rh\frac{f_1-f_0}{x_1-x_0} + rh(r-1)h\,\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0} + \cdots$$

$$f_n = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 \ldots + \frac{r(r-1)(r-2)}{n!}\Delta^n f_0 \ldots\ldots.. (1)$$

Since $x = x_0 - rh$ then $f_n(x) = f_n(x_0 - rh)$

To find the first and second derivatives of equation (1) then , derive equation (1) with respect to (r) gives

$$h\bar{f}_n(x_0 - rh) = \Delta f_0 + \frac{(2r-1)}{2!}\Delta^2 f_0 + \frac{3r^2-6r+2}{3!}\Delta^3 f_0 \ldots$$

$$\rightarrow \bar{f}_n(x_0 - rh) = \frac{1}{h}\left(\Delta f_0 + \frac{(2r-1)}{2}\Delta^2 f_0 + \frac{3r^2-6r+2}{6}\Delta^3 f_0\right)$$

$$h^2\bar{\bar{f}}_n(x_0 - rh) = \Delta^2 f_0 + (r-1)\Delta^3 f_0$$

$$\rightarrow \bar{\bar{f}}_n(x_0 - rh) = \frac{1}{h^2}(\Delta^2 f_0 + (r-1)\Delta^3 f_0)$$

For backward interpolation

$\nabla f_i = f_i - f_{i-1} \rightarrow \nabla^2 = \nabla f_i - \nabla f_{i-1}$ and so on $\Delta^n = \Delta^{n-1} f_i - \Delta^{n-1} f_{i-1}$

$f_n = f_0 + r\nabla f_0 + \frac{r(r-1)}{2!}\nabla^2 f_0 + \frac{r(r-1)(r-2)}{3!}\nabla^3 f_0 \ldots + \frac{r(r-1)(r-2)}{n!}\nabla^n f_0$ …….. (1)

Ex₁/ find three-point formula for the derivative of (f) at $x = x_0$ using forward interpolation

Sol:

Since there is three points then $(x_0, x_1, and\ x_2)$

x_i	f_i	Δf_i	$\Delta^2 f_i$
x_0	f_0		
		$f_1 - f_0$	
x_1	f_1		$f_2 - 2f_1 + f_0$
		$f_2 - f_1$	
x_2	f_2		

For $x = x_0$, this means that $(r = 0)$

$\bar{f}_n(x_0) = \frac{1}{h}\left(\Delta f_0 - \frac{1}{2}\Delta^2 f_0\right)$

$= \frac{1}{h}\left(f_1 - f_0 - \frac{1}{2} f_2 + f_1 - \frac{1}{2} f_0\right)$

$= \frac{1}{h}\left(2f_1 - \frac{3}{2} f_0 - \frac{1}{2} f_2\right)$

$= \frac{1}{2h}\left(-3f_0 + 4f_1 - f_2\right)$

Ex₂/ repeat example(1) for the second order and when $x = x_0$

Sol:

Since $\bar{\bar{f}}_n(x_0) = \frac{1}{h^2}(\Delta^2 f_0 - \Delta^3 f_0)$ for $x = x_0$, then

$\bar{\bar{f}}_n(x_0) = \frac{1}{h^2}(f_2 - 2f_1 + f_0)$

Note: in previous two examples it is found that when all the coefficients of $(f_0, f_1, and\ f_2)$ are added to each other, the result will be equal to zero.

Ex$_3$/ obtain forward difference for four-order interpolating polynomial

Sol:

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

Ex$_4$/ for $x = x_0$, find each of the following [Δf_0, $\Delta^2 f_0$, $\Delta^3 f_0$, $\Delta^4 f_0, \Delta f_1, \Delta^2 f_1, \Delta^3 f_1, \Delta f_2, \Delta^2 f_2$, and Δf_3] which obtained in example (3)

Sol:

$\Delta f_0 = f_1 - f_0$, $\Delta f_1 = f_2 - f_1, \Delta f_2 = f_3 - f_2, \Delta f_3 = f_4 - f_3$

$\Delta^2 f_0 = (f_2 - f_1) - (f_1 - f_0) \rightarrow \Delta^2 f_0 = (f_2 - 2f_1 + f_0)$

$\Delta^2 f_1 = (f_3 - f_2) - (f_2 - f_1) \rightarrow \Delta^2 f_1 = (f_3 - 2f_2 + f_1)$

$\Delta^3 f_0 = (\Delta^2 f_1 - \Delta^2 f_0) \rightarrow \Delta^3 f_0 = (f_3 - 2f_2 + f_1) - (f_2 - 2f_1 + f_0)$

$= (f_3 - 3f_2 + 3f_1 - f_0)$

$\Delta^2 f_2 = (f_4 - f_3) - (f_3 - f_2) \rightarrow \Delta^2 f_1 = (f_4 - 2f_3 + f_2)$

$\Delta^3 f_1 = (\Delta^2 f_2 - \Delta^2 f_1) \rightarrow \Delta^3 f_0 = (f_4 - 2f_3 + f_2) - (f_3 - 2f_2 + f_1)$

$= (f_4 - 3f_3 + 3f_2 - f_1)$

$\Delta^4 f_0 = \Delta^3 f_1 - \Delta^3 f_0 \rightarrow \Delta^4 f_0 = (f_4 - 3f_3 + 3f_2 - f_1) - (f_3 - 3f_2 + 3f_1 - f_0)$

$= (f_4 - 4f_3 + 5f_2 - 4f_1 + f_0)$

Ex$_4$/ find third order for $f(x) = x^2 + x + 1$

Sol:

$x_i \quad f_i \qquad\qquad \Delta f_i \qquad \Delta^2 f_i \qquad \Delta^3 f_i$

$x_0 \quad x^2 + x + 1$

$\qquad\qquad\qquad 2x + 2$

$x_1 \quad x^2 + 3x + 3 \qquad\qquad 2$

$\qquad\qquad\qquad 2x + 4 \qquad\qquad 0$

$x_2 \quad x^2 + 5x + 7 \qquad\qquad 2$

$\qquad\qquad\qquad 2x + 6$

$x_3 \quad x^2 + 7x + 13$

Ex$_5$/ use central derivative to find three point formula for the first derivative at $x = x_i$

Sol:

$x_i \qquad \Delta f_i \quad \Delta^2 f_i$

$x_0 \quad f_{i-1}$

$\qquad f_i - f_{i-1}$

$x_1 \quad f_i \qquad f_{i+1} - 2f_i + f_{i-1}$

$\qquad f_{i+1} - f_i$

$x_2 \quad f_{i+1}$

Since $\bar{f}_n(x_0 - rh) = \frac{1}{h}\left(\Delta f_0 + \frac{(2r-1)}{2}\Delta^2 f_0 + \frac{3r^2-6r+2}{6}\Delta^3 f_0\right)$

For $r = 1$

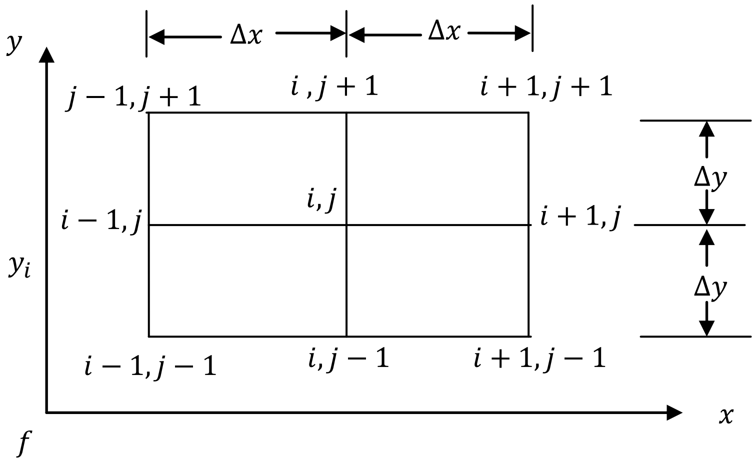
$\bar{f}_i = \frac{1}{h}\left(\Delta f_0 + \frac{1}{2}\Delta^2 f_0\right)$

$= \frac{1}{h}\left(f_i - f_{i-1} + \frac{1}{2}(f_{i+1} - 2f_i + f_{i-1})\right)$

$= \frac{1}{2h}(f_{i+1} - f_{i-1})$

❖ Mixed Derivatives

Assume that (f) is a function of two variables (x, y), in this case the mixed derivative will be $\frac{d^2 f}{dxdy}$, the central derivative approximations and a second order polynomial. Let the center at a point (x_i, y_j) and Consider the following figure



$$\frac{d^2 f}{dxdy} = \frac{d}{dx}\left(\frac{df}{dy}\right)$$

$$= \frac{d}{dx}\left[\frac{1}{2\Delta y}\left(f_{i,j+1} - f_{i,j-1}\right)\right]$$

$$= \frac{1}{2\Delta y}\left[\frac{d}{dx}(f_{i,j+1}) - \frac{d}{dx}(f_{i,j-1})\right]$$

$$= \frac{1}{4\Delta x \Delta y}\left[(f_{i+1,j+1} - f_{i-1,j+1}) - (f_{i+1,j-1} - f_{i-1,j-1})\right]$$

$$= \frac{1}{4\Delta x \Delta y}\left[f_{i+1,j+1} + f_{i-1,j-1} - f_{i-1,j+1} - f_{i+1,j-1}\right]$$

Ex$_6$/ Approximate the mixed partial derivative $\frac{d^4 f}{dx^2 dy^2}$ at $x = 1, y = 1$ for the function $f(x, y) = x^3 y^3$, use central differences and a second- order polynomial approximation. Note that (h) is the same at $(x \; and \; y)$ directions.

Sol:

$$\frac{d^4 f}{dx^2 dy^2} = \frac{d^2}{dx^2}\left(\frac{d^2}{dy^2}\right)$$

$$= \frac{d^2}{dx^2}\left[\frac{d}{dy}\left(\frac{df}{dy}\right)\right]$$

$$=\frac{d^2}{dx^2}\left[\frac{d}{dy}\left(\frac{1}{2\Delta y}\left(f_{i,j+1} - f_{i,j-1}\right)\right)\right]$$

$$=\frac{d^2}{dx^2}\left[\frac{1}{2\Delta y}\left(\frac{df_{i,j+1}}{dy} - \frac{df_{i,j-1}}{dy}\right)\right]$$

$$= \frac{d^2}{dx^2}\left[\frac{1}{2\Delta y}\left[\frac{1}{2\Delta y}\right]\left(f_{i,j+1} - f_{i,j} - (f_{i,j} - f_{i,j-1})\right)\right]$$, let $h = 2\Delta y$ then

$$= \frac{d^2}{dx^2}\left[\frac{1}{h^2}\left(f_{i,j+1} - 2f_{i,j} + f_{i,j-1}\right)\right]$$

$$= \frac{1}{h^2}\left[\frac{d}{dx}\left[\frac{df_{i,j+1}}{dx} - 2\frac{df_{i,j}}{dx} + \frac{df_{i,j-1}}{dx}\right]\right]$$

$=\frac{1}{h^2}\left[\frac{d}{dx}\left(\frac{1}{2\Delta x}\right)\left[(f_{i+1,j+1} - f_{i-1,j+1}) - 2(f_{i+1,j} - f_{i-1,j}) + (f_{i+1,j-1} - f_{i-1,j-1})\right]\right]$, let $2\Delta x = h$ then

$$=\frac{1}{h^4}\left[f_{i+1,j+1} - f_{i,j+1} - f_{i,j+1} - f_{i-1,j+1} - 2f_{i+1,j} + 2f_{i,j} + 2f_{i,j} - 2f_{i-1,j} - f_{i+1,j-1} - f_{i,j-1} + f_{i-1,j-1} - f_{i,j-1}\right]$$

$$\frac{1}{h^4}\left[f_{i+1,j+1} - 2f_{i,j+1} + f_{i-1,j+1} - 2f_{i+1,j} + 4f_{i,j} - 2f_{i-1,j} + f_{i+1,j-1} - 2f_{i,j-1} + f_{i-1,j-1}\right]$$

❖ ***Stencil Representation of Derivatives***

For equally spaced base points the difference approximations for the first and the second derivatives obtained earlier can be conveniently expressed in so- called "stencil" form. Consider foe example the equation of the first derivative that obtained in example one which represent the 1^{st}order forward derivative approximation

$\bar{f}_0 = \frac{1}{2h}\left(-3f_0 + 4f_1 - f_2\right) \rightarrow \bar{f}_0 = \frac{1}{2h}$((-3)—(4)—(-1))

Then $\bar{f}_0 = \frac{1}{2h}$

The double circles represent the position of the base point x_i.

The 1^{st}order central derivative approximation is given as

$\bar{f}_i = \frac{1}{2h}\ (f_{i+1} - f_{i-1})$

Can be represented as [$\frac{1}{2h}$((-1)—((0))—(1))

Ex$_7$/ represent the second order forward derivative approximation in stencil form

Sol:

Since $\bar{\bar{f}}_n(x_0) = \frac{1}{h^2}(f_2 - 2f_1 + f_0\ )$

Then the stencil form $\bar{\bar{f}}_n(x_0) = \frac{1}{h^2}$((1)—((-2))—(1))

Ex$_8$/ give the stencil form of the second order mixed derivative

Sol: