1- The Uniform Plane Waves:-

In this chapter we shall apply Maxweell's equation to introduce the fundamental theory of wave motion.

1-1 The wave equation:-

the effect of propagation of EM wave in medium : Free space ; Lossy dielectric ; Lossless dielectric (perfect dielectric) and Conducting media. These propagation phenomena for a type traveling wave called plane wave can be explained or derived by Maxwell's equations. Asumme that EM field exists in ahomogenous linear medium with parameter ε, μ & σ.

From Maxwell's equations :

$$\nabla\times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = -\mu \frac{\partial \overline{H}}{\partial t} \qquad (1)$$

$$\nabla\times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} = \overline{J} + \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad (2)$$

$$\nabla\bullet \overline{D} = \rho_v \qquad (3)$$

$$\nabla\bullet \overline{B} = 0 \qquad (4)$$

To eliminta H from the eq.(1) , let us apply the curl operator to the eq.(1):-

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}. \qquad (5)$$

In a similar manner we eliminate **E** from the first equation of the second pair, to obtain

$$\nabla \times (\nabla \times \mathbf{H}) = -\mu\sigma \frac{\partial \mathbf{H}}{\partial t} - \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2}. \qquad (6)$$

We know from vector analysis that for any vector function **F** that can be differentiated twice, $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

So eq.(5) & (6) become:-

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} = 0, \quad \textit{(Wave equation for vector } \mathbf{E}) \qquad (7)$$

$$\nabla^2 \mathbf{H} - \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} - \mu\sigma \frac{\partial \mathbf{H}}{\partial t} = 0. \quad \textit{(Wave equation for vector } \mathbf{H}) \qquad (8)$$

These are the *wave equations.*

If the field is time-harmonic and we use complex notation, we obtain

$$\nabla^2\underline{\mathbf{E}} + (\omega^2\epsilon\mu - j\omega\mu\sigma)\underline{\mathbf{E}} = 0,$$

[Helmholtz equation (complex wave equation) for vector **E***]*

$$\nabla^2\underline{\mathbf{H}} + (\omega^2\epsilon\mu - j\omega\mu\sigma)\underline{\mathbf{H}} = 0.$$

[Helmholtz equation (complex wave equation) for vector **H***]*

Where :-

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$= -\omega^2\mu\varepsilon + j\omega\mu\sigma$$

$\gamma =$ propagation constant $\gamma = \alpha + j\beta$

$$\gamma^2 = (\alpha^2 - \beta^2) + 2j\alpha\beta$$

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \quad \& \; 2\alpha\beta = \omega\mu\sigma$$

α:- is known as attenuation constant as a measure of the wave is attenuated while traveling in a medium .

β :- is phase constant

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right]} \quad rad/m \qquad \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right]} \; Np/m$$

Wave velocity $v_p = \frac{\omega}{\beta}$

1-2 Plane Wave In Lossless (Perfect) Dielectrics:-

Parameters :- $\sigma = 0, \varepsilon = \varepsilon_0\varepsilon_r, \mu = \mu_0\mu_r$

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\varepsilon}$$

$$v = u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \quad \lambda = \frac{2\pi}{\beta}$$

1-3 Plane Wave In Free Space:-

Free space is nothing more than the perfect dielectric media :

$$\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$$

$$\alpha = 0 \ , \quad \beta = \omega\sqrt{\mu_0 \varepsilon_0} = \omega / c$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c \ , \quad \lambda = \frac{2\pi}{\beta}$$

$$u = c \approx 3 \times 10^8 \ m/s$$

2- Radio Channel

Signal transmission in a radio system is based on converting the electrical signals generated by the transmitter into electromagnetic waves, propagation of waves in space and conversion back into electrical signals at the receiver side of the system.

The channel is a medium through which the transmitter output is sent. The physical medium between antennas where electromagnetic waves are propagated is called the **propagation channel**. Wireless communications systems exploit the air as a transmission medium. Obstacles such as buildings, hills and mountains influencing the propagation of electromagnetic waves, are considered to be part of the propagation channel. Hence, the propagation path from a radio transmitter antenna to a receiver antenna is called the radio **propagation channel**.

The propagation channel, together with **the antennas**, constitutes the **radio channel** as shown in Figure (1). The communication channel is described in terms of the **bandwidth** which determining the maximum information rate of the channel. The channel can introduce various kinds of distortion and delay into the signal as we will see later.

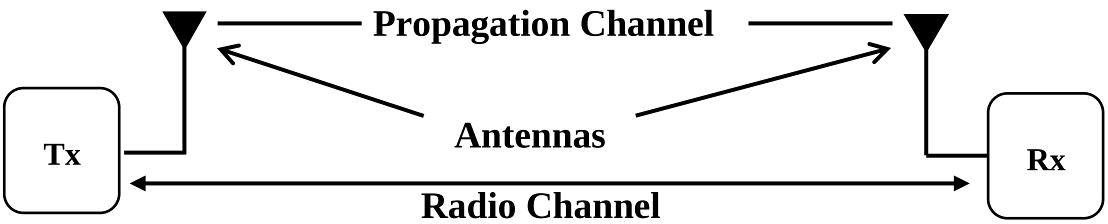
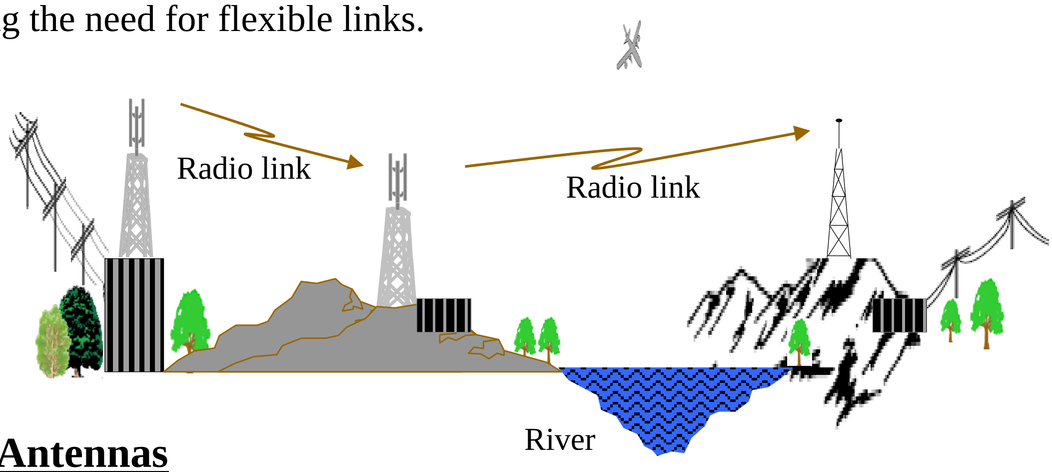


Figure (1) Radio and Propagation Channels

Advantages of the radio system include:

1- Operate over large distances.
2- Decrease the installation costs.
3- Be installed faster than other communication mediums.
4- Be easily re-installed in new locations as when plants are relocated.
5- Enable the user (subscriber) to transmit any information in any format that he requires.

6- Span lakes or rivers, where a cable facility would require special treatment to prevent water seepage onto the copper conductors, as shown in Figure 1
7- Overcome transmission obstacles such as mountains and deep valleys, where cable installation cost is too high and difficult to maintain.
8- Bypass the basic interconnection to the local telephone provider as shown in Figure 2.
Wireless communication is being seen as an alternative to quickly and cost effectively meeting the need for flexible links.



3- Antennas

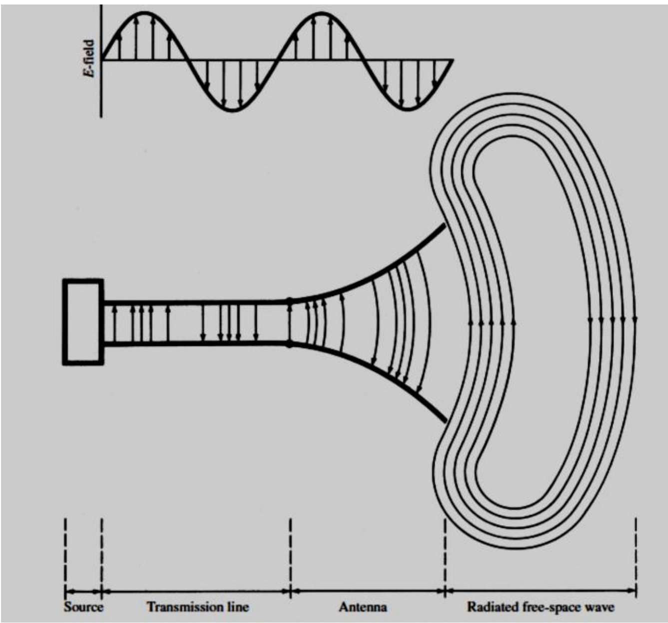
3-1 Fundamental parameters of Antennas:-

An antenna is an electrical conductor or system of conductors :

- Transmission – radiates EM energy into space.
- Reception – collects EM energy from space.
- In two ways commuincation, the same antenna can be used for transmission and reception .

An antenna is a curcuit element that provides atransition form a guided wave on transmission line to afree space wave and it provides for collection EM energy.

- In transmit system the RF signal is generated, amplified, modulated and applied to the antenna.
- In receive system the antenna collects Em waves.



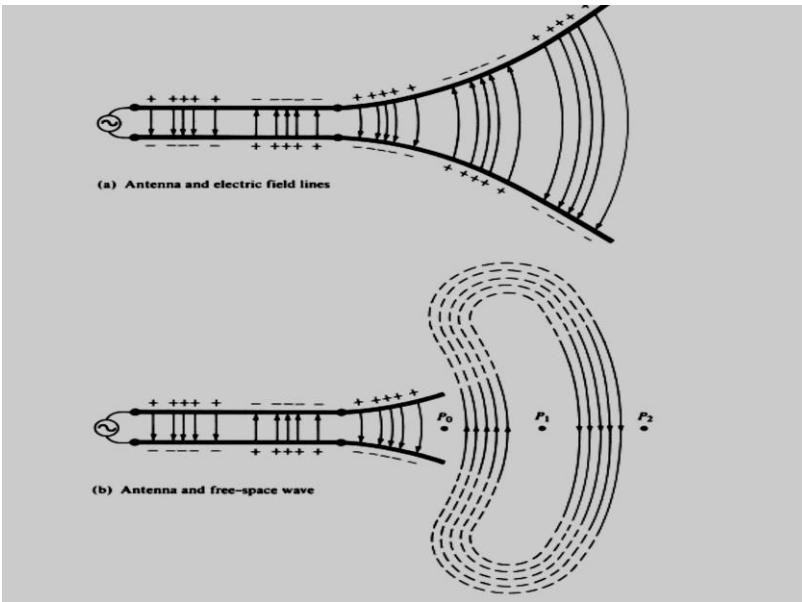


Figure 1.11 Source, transmission line, antenna, and detachment of electric field lines.

3-2 The ideal dipole:-

The simplest infinitesimal radiating element, called a *Hertzian dipole*, is a current element of length dl carrying I(t) amperes. Conservation of charge requires charge reservoirs at each end of the current element containing ±q(t) coulombs, where I = dq/dt. The total charge is zero.

Analysis of the Radiative Field

(other terms used for ideal dipole are hertzian electric dipole, electric dipole, infinitesimal dipole, and doublet).

- Very small finite length (dl << λ).
- Assume an element of current of length dl along z- axis.
- dl centered on the coordinate origin.
- Constant uinform amplitude current I.

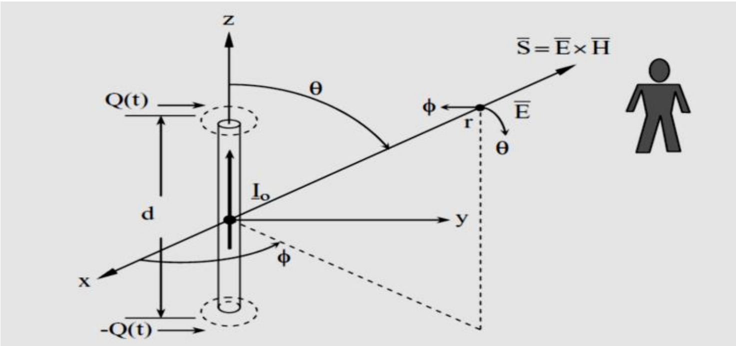


Figure (1): An elementary doublet.

3-2-1 Magnetic potential at a distance r from this current element is

$$A = \mu_0 \int \frac{i\,dl}{4\pi r}$$

The current in the doublet oscillatory with frequency $\omega = 2\pi f$

$$.i = I_0 \sin \omega t$$

The current is said to be retarded at point P because there is a propagation time delay r/c or phase delay βr from 0 to P, so that we may write:-

$$i = I_0 \sin \omega \left(t - \frac{r}{c} \right)$$

$$A = \mu_0 \int \frac{I_0 \sin \omega \left(t - \frac{r}{c} \right) dl}{4\pi r}$$

Referring to figure above, current element *dl* is symmetrically placed about the origin with its axis along the z axis, the current on the element has only a z-component, *A* also has only the A_z component. Thus, the magnetic potential at *P is*

$$A_z = \frac{\mu_0}{4\pi}\int\frac{I_0 \sin\omega\left(t-\frac{r}{c}\right)dl}{r}$$

$$\int_0^{dl} I dl = I dl$$

$$A_z = \frac{\mu_0}{4\pi}\frac{I_0\, dl \sin\omega\left(t-\frac{r}{c}\right)}{r}$$

Referring to figure (1) again, it is clear that A has only z-component and

$$A_x = A_y = 0.$$

Using the transformation between rectangular and spherical components is:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$A_r = A_z\cos\theta = \frac{\mu_0 I_0\, dl}{4\pi r}\sin\omega\left(t-\frac{r}{c}\right)\cos\theta$$

$$A_\theta = -A_z\sin\theta = -\frac{\mu_0 I_0\, dl}{4\pi r}\sin\omega\left(t-\frac{r}{c}\right)\sin\theta$$

$$A_\phi = 0$$

3-2-2 The magnetic field intensity H can be expressed as:-

$$H = \frac{1}{\mu_0}(\nabla\times A)$$

Thus, using curl equation from the Appendix-IV

$$H_r a_r + H_\theta a_\theta + H_\phi a_\phi = \frac{1}{\mu_0}\begin{vmatrix} \frac{1}{r^2\sin\theta}a_r & \frac{1}{r\sin\theta}a_\theta & \frac{1}{r}a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & r A_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Equating the coefficients of the a_r, a_θ and a_ϕ, gets

$$H_r = \frac{1}{\mu_0 r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$H_\theta = \frac{1}{\mu_0}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]$$

and
$$H_\phi = \frac{1}{\mu_0 r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

Since $A_\phi = 0$ and A_θ and A_r are not the function of ϕ. Hence

$$H_r = 0; \quad H_\theta = 0$$

$$H_\phi = \frac{1}{\mu_0 r}\left[\frac{\partial}{\partial r}\left(-\frac{\mu_0 I_0 dl}{4\pi}\sin\omega\left(t-\frac{r}{c}\right)\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{\mu_0 I_0 dl}{4\pi r}\sin\omega\left(t-\frac{r}{c}\right)\cos\theta\right)\right]$$

$$\boxed{H_\phi = \frac{I_0 dl\sin\theta}{4\pi}\left[\frac{\sin\omega\left(t-\frac{r}{c}\right)}{r^2} + \frac{\omega}{rc}\cos\omega\left(t-\frac{r}{c}\right)\right]}$$

There are two parts of which:

1) $\frac{1}{r^2}\sin\omega\,(t - {}^{r}/_{c})$ **:** Which represent the induction field.

2) $\frac{\omega}{rc}\cos\omega(t - {}^{r}/_{c})$: Represents the radiative field which is available at the point r. It is a progressive waveform and is the electromagnetic energy propagated in space. We generally take only this term into account for radiation.

3-2-3 <u>The electric field E can be obtaining from magnetic field i H by applying Maxwell's first equation for free space:</u>

$$\nabla \times H = \in \frac{\partial E}{\partial t} \qquad \text{(as } J = 0 \text{ for free space)}$$

The above equation in its components form can be written as follows :

$$\begin{vmatrix} \frac{1}{r^2\sin\theta} & \frac{1}{r\sin\theta}a_\theta & \frac{1}{r}a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\, H_\phi \end{vmatrix} = \in\left(\frac{\partial E_r}{\partial t} + \frac{\partial E_\theta}{\partial t} + \frac{\partial E_\phi}{\partial t}\right)$$

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left\{\frac{I_0 dl\sin^2\theta}{4\pi}\left(\frac{1}{r^2}\sin\omega t'+\frac{\omega}{rc}\cos\omega t'\right)\right\}\right]=\in\frac{\partial E_r}{\partial t}$$

where $t'=\left(t-\frac{r}{c}\right)$

$$\frac{1}{r\sin\theta}\left[\frac{I_0 dl}{4\pi}\left\{\frac{\sin\omega t'}{r^2}+\frac{\omega\cos\omega t'}{cr}\right\}\cdot\frac{\partial}{\partial\theta}(\sin^2\theta)\right]=\in\frac{\partial E_r}{\partial t}$$

$$\frac{1}{r\sin\theta}\left[\frac{I_0 dl}{4\pi}\left\{\frac{\sin\omega t'}{r^2}+\frac{\omega\cos\omega t'}{cr}\right\}\cdot 2\sin\theta\cdot\cos\theta\right]=\in\frac{\partial E_r}{\partial t}$$

$$\frac{2I_0 dl\cos\theta}{4\pi\in}\cdot\left[\frac{\sin\omega t'}{r^3}+\frac{\omega\cos\omega t'}{cr^2}\right]\partial t=\partial E_r$$

Taking the integral of the above equation, we yields

$$\frac{2I_0 dl\cos\theta}{4\pi\varepsilon}\int\left(\frac{\sin\omega t'}{r^3}+\frac{\omega\cos\omega t'}{cr^2}\right)dt=\int\partial E_r$$

$$E_r=\frac{2I_0\,dl\cos\theta}{4\pi\varepsilon}\left[-\frac{\cos\omega t'}{\omega r^3}+\frac{\omega\sin\omega t'}{\omega cr^2}\right]$$

$$\boxed{E_r=\frac{2I_0\,dl\cos\theta}{4\pi\varepsilon}\left[\frac{\sin\omega t'}{cr^2}-\frac{\cos\omega t'}{\omega r^3}\right]}$$

The electric field strength from a current element along the radial direction for $r >> \lambda$ is negligible because varying as a function of $1/r^2$ and $1/r^3$. Hence substituting H in equation (1.11) and performing similar operation as before, we obtain:-

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{I_0 dl \sin\theta}{4\pi}\left(\frac{\sin\omega t'}{r^2}+\frac{\omega\cos\omega t'}{rc}\right)r\right] = \epsilon\frac{\partial E_\theta}{\partial t}$$

$$-\frac{I_0 dl \sin\theta}{4\pi r}\left[\frac{\partial}{\partial r}\left(\frac{\sin\omega t'}{r}+\frac{\omega\cos\omega t'}{c}\right)\right] = \epsilon\frac{\partial E_\theta}{\partial t}$$

$$\frac{\partial E_\theta}{\partial t} = -\frac{I_0 dl \sin\theta}{4\pi r\varepsilon}\left[-\frac{\omega}{c}\sin\omega t'\left(-\frac{\omega}{c}\right)-\frac{\omega\cos\omega t'}{cr}-\frac{\sin\omega t'}{r^2}\right]$$

$$\partial E_\theta = -\frac{I_0 dl \sin\theta}{4\pi r\varepsilon}\left[-\frac{\omega^2\sin\omega t'}{c^2}+\frac{\omega\cos\omega t'}{cr}+\frac{\sin\omega t'}{r^2}\right]\partial t$$

Taking the integral of the above equation

$$\int\partial E_\theta = -\int\frac{I_0\, dl \sin\theta}{4\pi\varepsilon r}\left[-\frac{\omega^2}{c^2}\sin\omega t'+\frac{\omega}{c^2}\cos\omega t'+\frac{\sin\omega t'}{r^2}\right]dt$$

$$E_\theta = -\frac{I_0\, dl \sin\theta}{4\pi\varepsilon}\left[+\frac{\omega^2\cos\omega t'}{c^2\omega r}+\frac{\omega\sin\omega t'}{cr^2\,\omega}-\frac{\cos\omega t'}{r^3\omega}\right]$$

$$E_\theta = \frac{I_0\, dl \sin\theta}{4\pi\varepsilon}\left[\frac{\omega\cos\omega t'}{c^2 r}+\frac{\sin\omega t'}{cr^2}-\frac{\cos\omega t'}{\omega r^3}\right]$$

The inverse distance (1/r) first, term is the radiating field for $r \gg \lambda$. Thus,

$$E_\theta\ (\text{Radiation}) = \left(\frac{\omega I_0\, dl \sin\theta}{4\pi\varepsilon c^2 r}\right)\cos\omega\left(t-\frac{r}{c}\right)$$

$$H_\phi(\text{Radiation}) = \frac{\omega I_0\, dl \sin\theta}{4\pi cr}\cos\omega\left(t-\frac{r}{c}\right)$$

The equations by using phase delay βr from 0 to P:

$$H_\emptyset = \frac{I\Delta z}{4\pi}\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)e^{-j\beta r}\sin\theta$$

$$E = \frac{I\Delta z}{4\pi}\left[\frac{jw\mu}{r} + \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{r^2} + \frac{1}{jw\varepsilon r^3}\right]e^{-j\beta r}\sin\theta\,\hat{\theta} + \frac{I\Delta z}{2\pi}\left[\sqrt{\frac{\mu}{\varepsilon}}\frac{1}{r^2} + \frac{1}{jw\varepsilon r^3}\right]e^{-j\beta r}cos\theta\hat{r}$$

Note that if the medium surrounding the dipole is air or free space, β=$w\sqrt{\mu_0\varepsilon_0}$ can be written as:-

$$H = \frac{Idl}{4\pi}j\beta\left(1 + \frac{1}{j\beta r}\right)\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\emptyset}$$

$$E = \frac{Idl}{4\pi}jw\mu\left[1 + \frac{1}{j\beta r} + \frac{1}{(j\beta r)^2}\right]\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\theta} + \frac{Idl}{2\pi}\eta\left[\frac{1}{r} - j\frac{1}{\beta 2}\right]e^{-j\beta r}cos\theta\hat{r}$$

Radiation field term

Static field(quasi-stationary)

Induction field term

- For $r < \frac{\lambda}{2\pi}$, *the static field is dominant (region very close to the hertzian dipole)*
- For $r > \frac{\lambda}{2\pi}$, *the radiation field is dominant(far field).*
- For $r = \frac{\lambda}{2\pi}$, *the induction field is dominant(Near field).*
- If βr >> 1, far-field boundary condition, then Radiation field:-

$$H_{\emptyset} = \frac{Idl}{4\pi}j\beta\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\emptyset}$$

$$E_{\theta} = \frac{Idl}{4\pi}jw\mu\frac{e^{-j\beta r}}{r}\sin\theta\,\hat{\theta}$$

More specifically, we define the boundary between the near and the far zones by the value of r given by

$$r = \frac{2d^2}{\lambda}$$

where d is the largest dimension of the antenna.

So, a current element located in space, carrying an oscillatory current gives rise to two radiating field E_θ and $H_\emptyset$ the electric field being along the meridian (longitude) of the sphere (0) and the magnetic field along the latitude of the sphere (Ø). Both the fields are perpendicular to each other. By these, following results are obtained:

1.The field is maximum at the equator and equal to zero at poles.

2.The field intensities increase with frequency.

3.Both the fields are in time phase indicating transfer of energy.

Impedance is the ratio of the (E_θ) to magnetic (H_ϕ) :-

$$\eta = \frac{E_\theta}{H_\emptyset} = \frac{w\mu}{\beta} = \frac{w\mu}{w\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\eta_0 = 120\pi\ \Omega = 376.7\ \Omega\ for\ free\ space\ \ at\ \mu_0 = 4\pi \times 10^{-7} \&\ \varepsilon_0 = 8.854 \times 10^{-12}$$

So, the radiating wave has the same impedance as the free space impedance of 120 π ohms and hence the radiating field is matched into the free space impedance for maximum transfer of power.

3-2-4 Power radiated by a current element:-

The time-average power density:-

$$p_{ave} = \frac{1}{2} Re(E_s \times H_s) = \frac{1}{2} Re\left(E_{\theta s} H_{\emptyset}^{*} a_r\right)$$

This is the power radiated by the current element along the radius vector r of the sphere and is perpendicular to (φ) and (θ). The total power radiated is obtained by taking the surface integral of over any surface enclosing the current element as:-

$$p_{rad} = \int p_{ave}.ds = \frac{1}{2}\left(\frac{Idl}{4\pi}\right)^2 \int jw\mu \frac{e^{-j\beta r}}{r} \sin\theta\ \hat{\theta} \times -j\beta \frac{e^{-j\beta r}}{r} \sin\theta\ \widehat{\emptyset}$$

Then $\quad ds = r^2 \sin\theta\ d\theta\ d\emptyset\ \hat{r} \qquad when \qquad \widehat{\theta} \times \widehat{\emptyset} = \hat{r}$

$$p_{rad} = \frac{1}{2}\left(\frac{Idl}{4\pi}\right)^2 w\mu\beta \int_0^{2\pi} \int_0^{\pi} \frac{sin^2\theta}{r^2}\ r^2 \sin\theta\ d\theta\ d\emptyset$$

$$= \frac{1}{2}\left(\frac{Idl}{4\pi}\right)^2 w\mu\beta\ (2\pi)\frac{4}{3} = \frac{w\mu\beta}{12\pi}\ (Idl) = \ 40\ \pi^2\left(\frac{Idl}{\lambda}\right)^2$$

3-2-5 the radiation resistance:-

the power radiated from an ideal dipole of length dl <<λ and input current I is:-

$$p_{rad} = \frac{1}{2}\ R_r |I|^2$$

$$note\ that\ a\ factor \frac{1}{2} is\ present\ because\ current\ I\ is\ the\ peak\ value\ in\ the\ time\ waveform$$

$$p_{rad} = \frac{\omega\mu\beta}{12\pi}(Idl)^2 = \frac{1}{2}R_r|I|^2$$

Then for ideal dipole $9\,\frac{\omega\mu\beta}{12\pi}(dl)^2 = R_r = 80\pi^2(\frac{dl}{\lambda})^2\Omega$

Appendix

A-phscial constant

quantity	symbol	value	units
speed of light in vacuum	c_0, c	299 792 458	$m\,s^{-1}$
permittivity of vacuum	ϵ_0	$8.854\,187\,817 \times 10^{-12}$	$F\,m^{-1}$
permeability of vacuum	μ_0	$4\pi \times 10^{-7}$	$H\,m^{-1}$
characteristic impedance	η_0, Z_0	376.730 313 461	Ω
electron charge	e	$1.602\,176\,462 \times 10^{-19}$	C
electron mass	m_e	$9.109\,381\,887 \times 10^{-31}$	kg
Boltzmann constant	k	$1.380\,650\,324 \times 10^{-23}$	$J\,K^{-1}$
Avogadro constant	N_A, L	$6.022\,141\,994 \times 10^{23}$	mol^{-1}
Planck constant	h	$6.626\,068\,76 \times 10^{-34}$	J/Hz
Gravitational constant	G	$6.672\,59 \times 10^{-11}$	$m^3\,kg^{-1}s^{-2}$
Earth mass	$M_\oplus$	5.972×10^{24}	kg
Earth equatorial radius	a_e	6378	km

B-Electromagnetic Frequency Bands

RF Spectrum					
	band designations	frequency		wavelength	
ELF	Extremely Low Frequency	30–300	Hz	1–10	Mm
VF	Voice Frequency	300–3000	Hz	100–1000	km
VLF	Very Low Frequency	3–30	kHz	10–100	km
LF	Low Frequency	30–300	kHz	1–10	km
MF	Medium Frequency	300–3000	kHz	100–1000	m
HF	High Frequency	3–30	MHz	10–100	m
VHF	Very High Frequency	30–300	MHz	1–10	m
UHF	Ultra High Frequency	300–3000	MHz	10–100	cm
SHF	Super High Frequency	3–30	GHz	1–10	cm
EHF	Extremely High Frequency	30–300	GHz	1–10	mm
	Submillimeter	300-3000	GHz	100–1000	μm

C. Vector Identities and Integral Theorems

Algebraic Identities

$$|A|^2|B|^2 = |A\cdot B|^2 + |A\times B|^2 \tag{C.1}$$

$$(A\times B)\cdot C = (B\times C)\cdot A = (C\times A)\cdot B \tag{C.2}$$

$$A\times (B\times C) = B\,(A\cdot C) - C\,(A\cdot B) \qquad \text{(BAC-CAB rule)} \tag{C.3}$$

$$(A\times B)\cdot(C\times D) = (A\cdot C)\,(B\cdot D) - (A\cdot D)\,(B\cdot C) \tag{C.4}$$

$$(A\times B)\times(C\times D) = [\,(A\times B)\cdot D]\,C - [\,(A\times B)\cdot C]\,D \tag{C.5}$$

$$A = \hat{\mathbf{n}}\times(A\times\hat{\mathbf{n}}) + (\hat{\mathbf{n}}\cdot A)\,\hat{\mathbf{n}} = A_{\perp} + A_{\parallel} \tag{C.6}$$

where $\hat{\mathbf{n}}$ is any unit vector, and $A_{\perp}$, $A_{\parallel}$ are the components of A perpendicular and parallel to $\mathbf{n}$. Note also that $\hat{\mathbf{n}}\times(A\times\hat{\mathbf{n}}) = (\hat{\mathbf{n}}\times A)\times\hat{\mathbf{n}}$.

Differential Identities

$$\nabla\times(\nabla\psi) = 0 \tag{C.7}$$

$$\nabla\cdot(\nabla\times A) = 0 \tag{C.8}$$

$$\nabla\cdot(\psi A) = A\cdot\nabla\psi + \psi\nabla\cdot A \tag{C.9}$$

$$\nabla\times(\psi A) = \psi\nabla\times A + \nabla\psi\times A \tag{C.10}$$

$$\nabla(A\cdot B) = (A\cdot\nabla)B + (B\cdot\nabla)A + A\times(\nabla\times B) + B\times(\nabla\times A) \tag{C.11}$$

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B) \tag{C.12}$$

$$\nabla\times(A\times B) = A(\nabla\cdot B) - B(\nabla\cdot A) + (B\cdot\nabla)A - (A\cdot\nabla)B \tag{C.13}$$

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A \tag{C.14}$$

$$A_x\nabla B_x + A_y\nabla B_y + A_z\nabla B_z = (A\cdot\nabla)B + A\times(\nabla\times B) \tag{C.15}$$

$$B_x\nabla A_x + B_y\nabla A_y + B_z\nabla A_z = (B\cdot\nabla)A + B\times(\nabla\times A) \tag{C.16}$$

$$(\hat{\mathbf{n}}\times\nabla)\times A = \hat{\mathbf{n}}\times(\nabla\times A) + (\hat{\mathbf{n}}\cdot\nabla)A - \hat{\mathbf{n}}(\nabla\cdot A) \tag{C.17}$$

$$\begin{aligned}\psi(\hat{\mathbf{n}}\cdot\nabla)E - E(\hat{\mathbf{n}}\cdot\nabla\psi) = {} & [(\hat{\mathbf{n}}\cdot\nabla)(\psi E) + \hat{\mathbf{n}}\times(\nabla\times(\psi E)) - \hat{\mathbf{n}}\nabla\cdot(\psi E)] \\ & + [\hat{\mathbf{n}}\psi\nabla\cdot E - (\hat{\mathbf{n}}\times E)\times\nabla\psi - \psi\hat{\mathbf{n}}\times(\nabla\times E) - (\hat{\mathbf{n}}\cdot E)\nabla\psi]\end{aligned} \tag{C.18}$$

With $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, and the unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$, we have:

$$\nabla r = \hat{\mathbf{r}}, \quad \nabla r^2 = 2\mathbf{r}, \quad \nabla\frac{1}{r} = -\frac{\hat{\mathbf{r}}}{r^2}, \quad \nabla\cdot\mathbf{r} = 3, \quad \nabla\times\mathbf{r} = 0, \quad \nabla\cdot\hat{\mathbf{r}} = \frac{2}{r} \qquad \text{(C.19)}$$

Integral Theorems for Closed Surfaces

The theorems involve a volume V surrounded by a closed surface S. The divergence or Gauss' theorem is:

$$\boxed{\int_V \nabla\cdot\mathbf{A}\,dV = \oint_S \mathbf{A}\cdot\hat{\mathbf{n}}\,dS} \qquad \text{(Gauss' divergence theorem)} \qquad \text{(C.20)}$$

where $\hat{\mathbf{n}}$ is the *outward* normal to the surface. Green's first and second identities are:

$$\int_V \left[\varphi\nabla^2\psi + \nabla\varphi\cdot\nabla\psi\right] dV = \oint_S \varphi\frac{\partial\psi}{\partial n}\,dS \qquad \text{(C.21)}$$

$$\int_V \left[\varphi\nabla^2\psi - \psi\nabla^2\varphi\right] dV = \oint_S \left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right) dS \qquad \text{(C.22)}$$

where $\dfrac{\partial}{\partial n} = \hat{\mathbf{n}}\cdot\nabla$ is the directional derivative along $\hat{\mathbf{n}}$. Some related theorems are:

$$\int_V \nabla^2\psi\,dV = \oint_S \hat{\mathbf{n}}\cdot\nabla\psi\,dS = \oint_S \frac{\partial\psi}{\partial n}\,dS \qquad \text{(C.23)}$$

$$\int_V \nabla\psi\,dV = \oint_S \psi\,\hat{\mathbf{n}}\,dS \qquad \text{(C.24)}$$

$$\int_V \nabla^2\mathbf{A}\,dV = \oint_S (\hat{\mathbf{n}}\cdot\nabla)\mathbf{A}\,dS = \oint_S \frac{\partial\mathbf{A}}{\partial n}\,dS \qquad \text{(C.25)}$$

$$\oint_S (\hat{\mathbf{n}}\times\nabla)\times\mathbf{A}\,dS = \oint_S \left[\hat{\mathbf{n}}\times(\nabla\times\mathbf{A}) + (\hat{\mathbf{n}}\cdot\nabla)\mathbf{A} - \hat{\mathbf{n}}(\nabla\cdot\mathbf{A})\right] dS = 0 \qquad \text{(C.26)}$$

$$\int_V \nabla\times\mathbf{A}\,dV = \oint_S \hat{\mathbf{n}}\times\mathbf{A}\,dS \qquad \text{(C.27)}$$

Using Eqs. (C.18) and (C.26), we find:

$$\oint_S \left(\psi\frac{\partial\mathbf{E}}{\partial n} - \mathbf{E}\frac{\partial\psi}{\partial n}\right) dS = - \oint_S \left[\hat{\mathbf{n}}\,\psi\,\nabla\cdot\mathbf{E} - (\hat{\mathbf{n}}\times\mathbf{E})\times\nabla\psi - \psi\,\hat{\mathbf{n}}\times(\nabla\times\mathbf{E}) - (\hat{\mathbf{n}}\cdot\mathbf{E})\,\nabla\psi\right] dS \qquad \text{(C.28)}$$

The vectorial forms of Green's identities are [605,602]:

$$\int_V (\nabla\times\mathbf{A}\cdot\nabla\times\mathbf{B} - \mathbf{A}\cdot\nabla\times\nabla\times\mathbf{B})\,dV = \oint_S \hat{\mathbf{n}}\cdot(\mathbf{A}\times\nabla\times\mathbf{B})\,dS \qquad \text{(C.29)}$$

$$\int_V (\mathbf{B}\cdot\nabla\times\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\nabla\times\mathbf{B})\,dV = \oint_S \hat{\mathbf{n}}\cdot(\mathbf{A}\times\nabla\times\mathbf{B} - \mathbf{B}\times\nabla\times\mathbf{A})\,dS \qquad \text{(C.30)}$$

Integral Theorems for Open Surfaces

Stokes' theorem involves an open surface S and its boundary contour C:

$$\int_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A}\, dS = \oint_C \mathbf{A} \cdot d\mathbf{l} \quad \text{(Stokes' theorem)} \tag{C.31}$$

where $d\mathbf{l}$ is the tangential path length around C. Some related theorems are:

$$\int_S [\psi\, \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} - (\hat{\mathbf{n}} \times \mathbf{A}) \cdot \nabla \psi]\, dS = \oint_C \psi \mathbf{A} \cdot d\mathbf{l} \tag{C.32}$$

$$\int_S [(\nabla \psi)\, \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} - ((\hat{\mathbf{n}} \times \mathbf{A}) \cdot \nabla) \nabla \psi]\, dS = \oint_C (\nabla \psi) \mathbf{A} \cdot d\mathbf{l} \tag{C.33}$$

$$\int_S \hat{\mathbf{n}} \times \nabla \psi\, dS = \oint_C \psi\, d\mathbf{l} \tag{C.34}$$

$$\int_S (\hat{\mathbf{n}} \times \nabla) \times \mathbf{A}\, dS = \int_S [\hat{\mathbf{n}} \times (\nabla \times \mathbf{A}) + (\hat{\mathbf{n}} \cdot \nabla)\mathbf{A} - \hat{\mathbf{n}}(\nabla \cdot \mathbf{A})]\, dS = \oint_C d\mathbf{l} \times \mathbf{A} \tag{C.35}$$

$$\int_S \hat{\mathbf{n}}\, dS = \frac{1}{2} \oint_C \mathbf{r} \times d\mathbf{l} \tag{C.36}$$

Eq. (C.36) is a special case of (C.35). Using Eqs. (C.18) and (C.35) we find:

$$\begin{aligned} &\int_S \left(\psi \frac{\partial \mathbf{E}}{\partial n} - \mathbf{E} \frac{\partial \psi}{\partial n}\right) dS + \oint_C \psi \mathbf{E} \times d\mathbf{l} = \\ &= \int_S [\hat{\mathbf{n}}\, \psi\, \nabla \cdot \mathbf{E} - (\hat{\mathbf{n}} \times \mathbf{E}) \times \nabla \psi - \psi\, \hat{\mathbf{n}} \times (\nabla \times \mathbf{E}) - (\hat{\mathbf{n}} \cdot \mathbf{E})\, \nabla \psi]\, dS \end{aligned} \tag{C.37}$$

Cartesian Coordinates

$$\nabla \psi = \hat{\mathbf{x}} \frac{\partial \psi}{\partial x} + \hat{\mathbf{y}} \frac{\partial \psi}{\partial y} + \hat{\mathbf{z}} \frac{\partial \psi}{\partial z}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial_x} & \dfrac{\partial}{\partial_y} & \dfrac{\partial}{\partial_z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical Coordinates

$$\nabla\psi = \hat{\boldsymbol{\rho}}\frac{\partial\psi}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{\mathbf{z}}\frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right)$$

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$$

Spherical Coordinates

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right) + \hat{\boldsymbol{\phi}}\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}') = \frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$$

$$\rho = r\sin\theta \qquad \hat{\mathbf{r}} = \hat{\mathbf{z}}\cos\theta + \hat{\boldsymbol{\rho}}\sin\theta \qquad \hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

$$z = r\cos\theta \qquad \hat{\boldsymbol{\theta}} = -\hat{\mathbf{z}}\sin\theta + \hat{\boldsymbol{\rho}}\cos\theta \qquad \hat{\boldsymbol{\rho}} = \hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$$

$$x = r\sin\theta\cos\phi \qquad \hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi\sin\theta + \hat{\mathbf{y}}\sin\phi\sin\theta + \hat{\mathbf{z}}\cos\theta$$

$$y = r\sin\theta\sin\phi \qquad \hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\phi\cos\theta + \hat{\mathbf{y}}\sin\phi\cos\theta - \hat{\mathbf{z}}\sin\theta$$

$$z = r\cos\theta \qquad \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

and the inverse relationships:

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$$

$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

Transformations Between Coordinate Systems

A vector **A** can be expressed component-wise in the three coordinate systems as:

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{x}}\,A_x + \hat{\mathbf{y}}\,A_y + \hat{\mathbf{z}}\,A_z \\ &= \hat{\boldsymbol{\rho}}\,A_\rho + \hat{\boldsymbol{\phi}}\,A_\phi + \hat{\mathbf{z}}\,A_z \\ &= \hat{\mathbf{r}}\,A_r + \hat{\boldsymbol{\theta}}\,A_\theta + \hat{\boldsymbol{\phi}}\,A_\phi\end{aligned} \tag{E.4}$$

The components in one coordinate system can be expressed in terms of the components of another by using the following relationships between the unit vectors, which were also given in Eqs. (13.8.1)–(13.8.3):

$$\begin{aligned} x &= \rho\cos\phi \qquad & \hat{\boldsymbol{\rho}} &= \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \qquad & \hat{\mathbf{x}} &= \hat{\boldsymbol{\rho}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \\ y &= \rho\sin\phi \qquad & \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \qquad & \hat{\mathbf{y}} &= \hat{\boldsymbol{\rho}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \end{aligned} \tag{E.5}$$