Discrete-time Fourier transform

In mathematics, the **discrete-time Fourier transform (DTFT)** is one of the specific forms of Fourier analysis. As such, it transforms one function into another, which is called the *frequency domain* representation, or simply the "DTFT", of the original function (which is often a function in the time-domain). But the DTFT requires an input function that is *discrete*. Such inputs are often created by sampling a continuous function, like a person's voice.

The DTFT frequency-domain representation is always a periodic function. Since one period of the function contains all of the unique information, it is sometimes convenient to say that the DTFT is a transform to a "finite" frequency-domain (the length of one period), rather than to the entire real line. It is Pontryagin dual to the Fourier series, which transforms from a periodic domain to a discrete domain.

Definition

Given a discrete set of real or complex numbers: $x[n],\ n \in \mathbb{Z}$(integers), the **discrete-time Fourier transform** (or **DTFT**) of $x[n]$is usually written:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n}. \qquad \textbf{(Eq.1)}$$

Relationship to sampling

Often the $x[n]$sequence represents the values (aka *samples*) of a continuous-time function, $x(t)$, at discrete moments in time: $t = nT$, where $T$is the sampling interval (in seconds), and $1/T = f_s$ is the sampling rate (samples per second). Then the DTFT provides an approximation of the continuous-time Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i2\pi ft}\, dt.$$

To understand this, consider the Poisson summation formula, which indicates that a periodic summation of function $X(f)$ can be constructed from the samples of function $x(t)$. The result is:

$$X_T(f) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} X(f - kf_s) \equiv T \sum_{n=-\infty}^{\infty} x(nT)\, e^{-i2\pi fTn}. \qquad \textbf{(Eq.2)}$$

The right-hand sides of **Eq.2** and **Eq.1** are identical with these associations:

$$x[n] = T \cdot x(nT)$$
$$\omega = 2\pi fT = 2\pi \left(\frac{f}{f_s}\right).$$

$X_T(f)$ Comprises exact copies of $X(f)$ that are shifted by multiples of f_s and combined by addition. For sufficiently large f_s, the $k=0$ term can be observed in the region $[-f_s/2, f_s/2]$ with little or no distortion (aliasing) from the other terms.

Normalized frequency

Since f represents ordinary frequency (*cycles per second*) and f_s has units of *samples per second*, the units of f/f_s are *cycles per sample*. It is common notational practice to replace this ratio with a single variable, called normalized frequency, which represents actual frequencies as multiples (usually fractional) of the sample rate. ω, as defined above, is also a normalized frequency, but its units are *radians per sample*. The normalized frequency has the added bonus that the function $X(\omega)$ is periodic with period 2π. So the inverse transform need only be evaluated in the interval 2π.

Periodicity

Sampling $x(t)$ causes its spectrum (DTFT) to become periodic. In terms of ordinary frequency, f (cycles per second), the period is the sample rate, f_s. In terms of normalized frequency, f/f_s (cycles per sample), the period is 1. And in terms of ω (radians per sample), the period is 2π, which also follows directly from the periodicity of $e^{-i\omega n}$. That is:

$$e^{-i(\omega+2\pi k)n} = e^{-i\omega n}$$

Where both n and k are arbitrary integers. Therefore:

$$X(\omega+2\pi k) = X(\omega)$$

The popular alternate notation $X(e^{i\omega})$ for the DTFT $X(\omega)$:

1. highlights the periodicity property, and
2. helps distinguish between the DTFT and underlying Fourier transform of $x(t)$; that is, $X(f)$ (or $X(\omega)$), and
3. Emphasizes the relationship of the DTFT to the Z-transform.

However, its relevance is obscured when the DTFT is formed by the frequency domain method (superposition), as discussed above. So the notation $X(\omega)$ is also commonly used, as in the table to follow.

Inverse transform

The following inverse transforms recover the discrete-time sequence:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cdot e^{i\omega n}\,d\omega$$
$$= T\int_{-\frac{1}{2T}}^{\frac{1}{2T}} X_T(f)\cdot e^{i2\pi fnT}\,df.$$

The integrals span one full period of the DTFT, which means that the x[n] samples are also the coefficients of a Fourier series expansion of the DTFT. Infinite limits of integration change the transform into a continuous-time Fourier transform [inverse], which produces a sequence of Dirac impulses. That is**:**

$$\int_{-\infty}^{\infty} X_T(f)\cdot e^{i2\pi ft}\,df = \int_{-\infty}^{\infty}\left(T\sum_{n=-\infty}^{\infty} x(nT)\,e^{-i2\pi fTn}\right)\cdot e^{i2\pi ft}\,df$$
$$= \sum_{n=-\infty}^{\infty} T\cdot x(nT)\int_{-\infty}^{\infty} e^{-i2\pi fTn}\cdot e^{i2\pi ft}\,df$$
$$= \sum_{n=-\infty}^{\infty} x[n]\cdot\delta(t-nT).$$

Finite-length sequences

For practical evaluation of the DTFT numerically, a finite-length sequence is obviously needed. For instance, a long sequence might be modified by a rectangular window function, resulting in**:**

$$X(\omega) = \sum_{n=0}^{L-1} x[n]\,e^{-i\omega n}$$, where L is the modified sequence length.

This is often a useful approximation of the spectrum of the unmodified sequence. The difference is a loss of clarity (resolution), which improves as L increases. It is common to evaluate *X*(ω) at an arbitrary number (*N*) of uniformly-spaced frequencies across one period (2π)**:**

$$\omega_k = \frac{2\pi}{N}k, \quad \text{for } k = 0, 1, \ldots, N-1$$

Which gives**?**

$$X[k] = X(\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-i2\pi\frac{k}{N}n}$$

When $N \geq L$, this can also be written**:**

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi\frac{k}{N}n},$$ because we define $x[n] = 0$ for $n \geq L$.

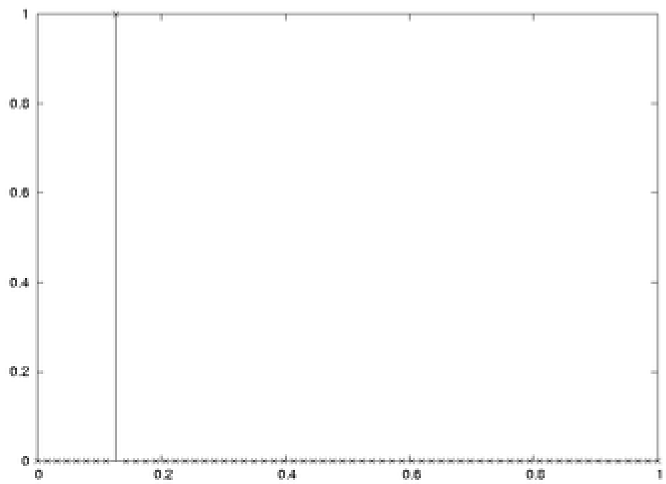
With that cosmetic adjustment, the $X[k]$ sequence is now recognizable as a discrete Fourier transform (DFT). While *N* defines the resolution at which we sample the DTFT, *L* limits the inherent resolution of the DTFT itself. So they are usually similar (or equal) values. And while it is common to choose *N* > *L*, the *only reason* to include the zero-valued terms in the summation is to take advantage of a fast Fourier transform algorithm for computing the DFT. However, when that is done it is often given undue significance, such as *zero-padded DFT* and/or *interpolated DFT*. But obviously exactly the same DFT can be calculated straightforwardly without the zero-valued terms. One can also compute the DTFT for the case of *N* < *L* (or for other frequency samplings) where it is not equivalent to a DFT.

To illustrate why *N* > *L* is common, consider the sequence**:**

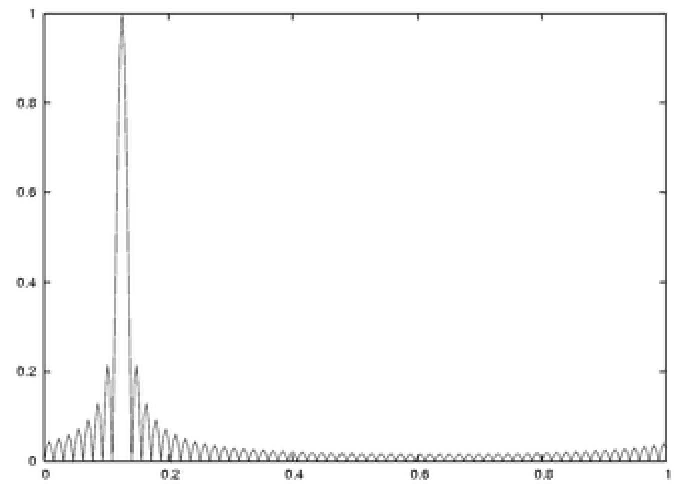
$x[n] = e^{i2\pi\frac{1}{8}n}$, and *L* = 64.

The two figures below are plots of the magnitude of two different sized DFTs, as indicated in their labels. In both cases, the dominant component is at the signal frequency**:** $f = \frac{1}{8} = 0.125$. Also visible on the right is the *spectral leakage* pattern of the

L = 64 rectangular window. The illusion on the left is a result of sampling the DTFT at all of its zero-crossings. Rather than the DTFT of a finite-length sequence, it gives the impression of an infinitely long sinusoidal sequence. Contributing factors to the illusion are the use of a rectangular window, and the choice of a frequency ($\frac{1}{8} = \frac{8}{64}$) with exactly 8 (an integer) cycles per 64 samples.



DFT for L = 64 and N = 64



DFT for L = 64 and N = 256

Difference between the DTFT and other Fourier transforms

The DTFT is the reverse of the Fourier series, in that the Fourier series has a continuous, periodic input and a discrete spectrum: formally, they are Pontryagin dual. The applications of the two transforms, however, are quite different.

The DFT and the DTFT can be viewed as the logical result of applying the standard continuous Fourier transform to discrete data. From that perspective, we have the satisfying result that it's not the transform that varies; it's just the form of the input:

- If it is discrete, the Fourier transform becomes a DTFT.

- If it is periodic, the Fourier transform becomes a Fourier series.
- If it is both, the Fourier transform becomes a DFT.

One can summarize this data in terms of the original domain and the transform domain:

Transform	Original domain	Transform domain
Fourier transform	**R**	**R**
Fourier series	S^1	**Z**
DTFT	**Z**	S^1
DFT	**Z**/*n***Z**	**Z**/*n***Z**

Where **R** is the real line, S^1 is the circle (the domain for periodic functions), **Z** is the integers (the domain for discrete functions), and **Z**/*n***Z** is the integers modulo *n* (as in modular arithmetic), the domain for periodic discrete functions.

From the point of view of Pontryagin duality, the Fourier transform and the DFT are self-dual, as the original domain and transform domain are isomorphic (the original domain and transform domain should be thought of as two separate copies of **R**, respectively **Z**/*n***Z**, not as the same space) while Fourier series and the DTFT are dual to each other.

Relationship to the Z-transform

The DTFT is a special case of the Z-transform. The bilateral Z-transform is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

So the special case is: $z = e^{i\omega}$. Since $|e^{i\omega}| = 1$, it is the evaluation of the Z-transform around the unit circle in the complex plane.

Table of discrete-time Fourier transforms

Some common transform pairs are shown below. The following notation applies:

- n is an integer representing the discrete-time domain (in samples)
- ω is a real number in $(-\pi, \pi)$, representing continuous angular frequency (in radians per sample).
  - The remainder of the transform $(|\omega| > \pi)$ is defined by: $X(\omega + 2\pi k) = X(\omega)$
- $u[n]$ is the discrete-time unit step function
- $\mathrm{sinc}(t)$ is the normalized sinc function
- $\delta(\omega)$ is the Dirac delta function
- $\delta[n]$ is the Kronecker delta $\delta_{n,0}$
- $\mathrm{rect}(t)$ is the rectangle function for arbitrary real-valued *t*:

$$\mathrm{rect}(t) = \sqcap(t) = \begin{cases} 0 & \text{if } |t| > \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 1 & \text{if } |t| < \frac{1}{2} \end{cases}$$

- $\mathrm{tri}(t)$ is the triangle function for arbitrary real-valued *t*:

$$\mathrm{tri}(t) = \wedge(t) = \begin{cases} 1+t; & -1 \le t \le 0 \\ 1-t; & 0 < t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Table: (see last pages)

Properties

This table shows the relationships between generic discrete-time Fourier transforms. We use the following notation:

- $*$ is the convolution between two signals
- $x[n]^*$ is the complex conjugate of the function *x[n]*
- $\rho_{xy}[n]$ represents the correlation between *x[n]* and *y[n]*.

The first column provides a description of the property; the second column shows the function in the time domain, the third column shows the spectrum in the frequency domain:

Property	Time domain $x[n]$	Frequency domain $X(\omega)$	Remarks
Linearity	$ax[n] + by[n]$	$aX(e^{i\omega}) + bY(e^{i\omega})$	
Shift in time	$x[n-k]$	$X(e^{i\omega})e^{-i\omega k}$	integer *k*
Shift in frequency (modulation)	$x[n]e^{ian}$	$X(e^{i(\omega-a)})$	real number *a*
Time reversal	$x[-n]$	$X(e^{-i\omega})$	
Time conjugation	$x[n]^*$	$X(e^{-i\omega})^*$	
Time reversal & conjugation	$x[-n]^*$	$X(e^{i\omega})^*$	
Derivative in frequency	$\frac{n}{i}x[n]$	$\frac{dX(e^{i\omega})}{d\omega}$	
Integral in frequency	$\frac{i}{n}x[n]$	$\int_{-\pi}^{\omega} X(e^{i\vartheta})d\vartheta$	
Convolve in time	$x[n] * y[n]$	$X(e^{i\omega}) \cdot Y(e^{i\omega})$	
Multiply in time	$x[n] \cdot y[n]$	$\frac{1}{2\pi}X(e^{i\omega}) * Y(e^{i\omega})$	
Correlation	$\rho_{xy}[n] = x[-n]^* * y[n]$	$R_{xy}(\omega) = X(e^{i\omega})^* \cdot Y(e^{i\omega})$	
Parseval's theorem	$E = \sum_{n=-\infty}^{\infty} x[n]y^*[n]$	$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})Y^*(e^{i\omega})d\omega$	

Symmetry Properties

The Fourier Transform can be decomposed into a real and imaginary part or into an even and odd part.

$$X(e^{i\omega}) = X_R(e^{i\omega}) + iX_I(e^{i\omega})$$

or

$$X(e^{i\omega}) = X_E(e^{i\omega}) + X_O(e^{i\omega})$$

Time Domain	Frequency Domain
$x[n]$	$X(e^{i\omega})$
$x^*[n]$	$X^*(e^{-i\omega})$
$x^*[-n]$	$X^*(e^{i\omega})$

Relations between **Fourier** ***transforms and Fourier series***

In the mathematical field of harmonic analysis, the continuous Fourier transform has very precise relations with Fourier series. It is also closely related to the discrete-time Fourier transform (DTFT) and the discrete Fourier transform (DFT).

The Fourier transform can be applied to time-discrete or time-periodic signals using the δ-Dirac formalism. In fact the Fourier series, the DTFT and the DFT can be derived all from the general continuous Fourier transform. They are, from a theoretical point of view, particular cases of the Fourier transform.

In signal theory and digital signal processing (DSP), the DFT (implemented as fast Fourier transform) is extensively used to calculate approximations to the spectrum of a continuous signal, knowing only a sequence of sampled points. The relations between DFT and Fourier transform are in this case essential.

Definitions

In the following table the definitions for the continuous Fourier transform, Fourier series, DTFT and DFT are reported:

Fourier transformations definitions

×	Continuous time	Discrete time
Time aperiodic	$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi ft}\, df$	$x[n] = \Delta t \int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} \bar{X}(f)\, e^{i2\pi fnT}\, df$
-	$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi ft}\, dt$	$\bar{X}(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-i2\pi fnT}$
Time periodic	$\bar{x}(t) = \sum_{k=-\infty}^{+\infty} X[k]\, e^{i\frac{2\pi k}{T_0}t}$	$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k\, e^{i\frac{2\pi}{N}kn} \qquad n = 0, \dots, N-1.$
-	$X[k] = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \bar{x}(t)\, e^{-i\frac{2\pi k}{T_0}t}\, dt$	$X_k = \sum_{n=0}^{N-1} x_n\, e^{-i\frac{2\pi}{N}kn} \qquad k = 0, \dots, N-1$

The table shows the properties of the time-domain signal:

- *Continuous time* versus *Discrete Time* (columns),
- *Aperiodic in time* versus *Periodic in time* (rows).

Equations needed to relate the various transformations

The definitions given in the previous section can be introduced axiomatically or can be derived from the continuous Fourier transform using the extend formalism of Dirac delta. Using this formalism the Continuous Fourier transform can be applied also to discrete or periodic signals.

To calculate the continuous Fourier transform of discrete and/or periodic signals we need to introduce some equations and recall some Fourier transform properties. Here is reported a list of them:

1. The **first** **Poisson summation formula**:

$$\sum_{n=-\infty}^{+\infty} x(t-nT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T_0}\right)\, e^{i\frac{2\pi k}{T_0}t}$$

2. The **second** **Poisson summation formula**:

$$\sum_{n=-\infty}^{+\infty} x(nT)\, e^{-i2\pi nfT} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(f-\frac{k}{T}\right)$$

3. The **Dirac comb** **transform** is important to understand the link between the continuous and the discrete or periodic case:

$$\sum_{n=-\infty}^{+\infty} \delta(t-nT) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{T}\sum_{k=-\infty}^{+\infty} \delta\left(f-\frac{k}{T}\right)$$

4. The theorems which define the Fourier transform properties, in particular the ***convolution*** **property**.

All these equations and properties can be demonstrated on their own.

Once calculated, the continuous Fourier transform of discrete and/or periodic signals can be related to the DTFT, the Fourier series and to the DFT definitions given above.

Relationship between the various transform

The following figure represents the relations between the various transforms.

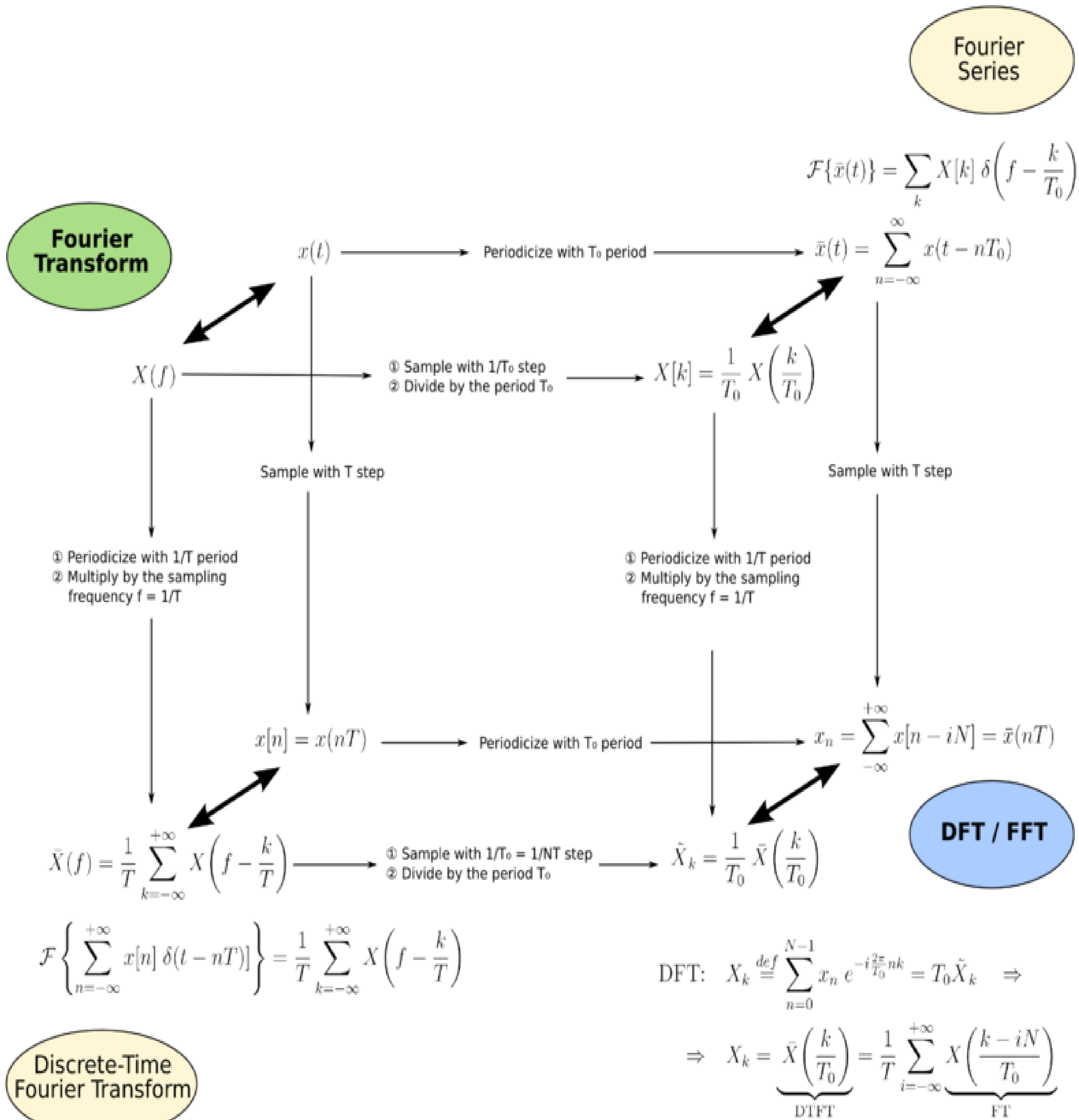


Figure 2. A "cube" graphs representing the relations between Fourier transform, DTFT, Fourier series and DFT. Each side of the cube indicates the operations needed to pass from one vertex to the other. The bolder double arrows indicate the link between each function in the direct domain and its transform.

Explanation of the symbols:

- The signal and its transform are bound by a bold double arrow ($\longleftrightarrow$)
- $x[n]$ and $X[k]$ are infinite sequences
- $\bar{x}(t)$ and $\bar{X}(f)$ are periodic functions
- x_n, X_k and $\tilde{X}_k$ are finite sequences
- $\mathcal{F}\{\dots\}$ indicates exclusively the *continuous Fourier transform*.

The *Poisson summation formulas* allow to link the Fourier series and the DTFT to the Fourier transform (respectively formula **1.** and **2.**).

The *convolution property* (**4.**) and *Dirac comb transform* (**3.**) allow to calculate the Fourier transform for the time-periodic or time-discrete signals as function of $X(f)$. In Figure 2 is showed what operations correspond, in the spectral domain, to the sampling of a continuous signal or to the periodicization of an aperiodic signal.

From Figure 2 we can see that the time domain sampling has the same effect on the spectrum both for an aperiodic signal ($x(t)$) and for a periodic signal ($\bar{x}(t)$). Conversely, the time domain periodicization has he same spectral effect on a continuous signal ($x(t)$) and on a discrete signal ($x[n]$).

DFT *versus* continuous Fourier transform.

The discrete Fourier transform (DFT) is the transform of a finite sequence. A finite sequence can be thought of as a time-periodic and time-discrete signal considered only in one period. For this reason the spectrum must be both periodic and discrete.

Following the Poisson formulas we would obtain $\tilde{X}_k$ as DFT definition. However, the DFT is defined usually as X_k (see Figure 2 or the previous definitions). For this reason the link between the DFT and the periodical transform $\bar{X}(f)$ is different by a scale factor from the relation obtained by the application of the Poisson formulas (which bring to $\tilde{X}_k$ and not to X_k).

Sample points of the spectrum of a continuous signal can be accurately calculated if the signal is band-limited and the sampling is done at a frequency above the Nyquist frequency. In this case, if the signal is time limited, we can begin sampling it before the signal "begins" and stop sampling after the signal "ends". Calculating the DFT of this finite sequence obtained from such sampling we obtain the sampled values of the spectrum of the original signal, apart a scale factor 1 / *T* (where *T* is the sampling step):

$$\underbrace{X_k}_{DFT} = \underbrace{\bar{X}\left(\frac{k}{NT}\right)}_{\text{DTFT}} = \frac{1}{T}\sum_{i=-\infty}^{+\infty} \underbrace{X\left(\frac{k-iN}{NT}\right)}_{\text{FT}} \simeq \frac{1}{T}\underbrace{X\left(\frac{k}{NT}\right)}_{\text{FT}} \qquad k = 0,\ldots,N-1$$

The last $\simeq$equality is between the periodic spectrum $\bar{X}(f)$evaluated in one period and the spectrum of the continuous signal *X*(*f*). The $\simeq$symbol is also used to stress that, if the signal is not perfectly band limited, we always get a bit of aliasing so the equality is not exact.

Usually in digital signal processing (DSP) the signal is too long to be analyzed as a whole. In this case windowing is used to calculate approximate spectrum samples of a small portion of the entire signal. This process inevitably adds further errors such **leakage** and **scalloping loss** (see Window function).

DTFT *versus* continuous Fourier transform

The Discrete-time Fourier transform (DTFT) is the transform of a discrete sequence. Since the time-domain is discrete the spectrum is periodic.

A discrete signal $x[n]$ can thought as the sampling of a continuous signal $x(t)$ with step T. The sampled signal can be treated as a continuous signal using the Dirac delta formalism. In particular the sampling operation is equivalent to the multiplication by a Dirac comb:

$$x_{\text{sampled}}(t) = x(t) \cdot \left(\sum_{n=-\infty}^{+\infty} \delta(t - nT) \right) = \sum_{n=-\infty}^{+\infty} x(t)\, \delta(t - nT) =$$
$$= \sum_{n=-\infty}^{+\infty} x(nT)\, \delta(t - nT) = \sum_{n=-\infty}^{+\infty} x[n]\, \delta(t - nT)$$

Calculating the Fourier transform of the sampled signal using the convolution property (**3.**) and the comb transform (**2.**), and then applying the second Poisson summation formula, we obtain:

$$\mathcal{F}\{x_{\text{sampled}}(t)\} = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(f - \frac{k}{T}\right) = \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-i2\pi n f T} = \bar{X}(f)$$

where $X(f)$ is the Fourier transform of the continuous signal $x(t)$. We see that the Fourier transform of $x_{\text{sampled}}(t)$ **is equal** to the DTFT of $x[n]$. The DTFT definition can be seen as formula to calculate the Fourier transform of the sampled signal using only the sampled values $x[n]$ (without the Dirac delta formalism). The last equation is reported in the lower left corner of Figure 2.

Another important aspect to note is that the *time-domain sampling* with step T corresponds to a periodicization of the spectrum with period $1 / T$ and a multiplication of the spectrum by an $1 / T$ factor. This relation can be seen in Figure 2 along the vertical arrows that go from $x(t)$ to $x[n]$ and from $X(f)$ to $\bar{X}(f)$.

Fourier series *versus* continuous Fourier transform

The Fourier series is an expansion of a periodic signal as a linear combination of discrete harmonic components. Since the signal is time-periodic the spectral components are not spread over a continuum range of frequency but are concentrated in discrete, equally spaced, frequency values. These discrete frequencies are all multiple of a base harmonic called **fundamental**. The fundamental harmonic is equal to the inverse of the period of the signal.

A periodic signal $\bar{x}(t)$can thought as the periodicization with period T_0 of an aperiodic signal $x(t)$. In particular, the periodicization is equivalent to the convolution ($*$ symbol) of $x(t)$ by a Dirac comb:

$$x_{\text{periodic}}(t) = \bar{x}(t) = x(t) * \sum_{n=-\infty}^{+\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

Calculating the Fourier transform of the periodic signal using the convolution property (**4.**) and the comb transform (**3.**), and then applying the first Poisson summation formula (**1.**), we obtain:

$$\mathcal{F}\{x_{\text{periodic}}\} \stackrel{3,4}{=} X(f) \cdot \frac{1}{T} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T}\right) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(\frac{k}{T}\right) \delta\left(f - \frac{k}{T}\right) =$$
$$\stackrel{1}{=} \sum_{k=-\infty}^{+\infty} X[k]\, \delta\left(f - \frac{k}{T}\right)$$

where $X(f)$ is the Fourier transform of the aperiodic signal $x(t)$, and $X[k]$ are the coefficients of the Fourier series expansion for the periodic signal $\bar{x}(t)$. This equation shows that the coefficients of the Fourier expansion of a periodic signal **are equal** to the amplitudes of the Dirac deltas of the Fourier transform. The last equation is reported in the upper right corner of Figure 2.

Another important aspect is that the time-domain periodicization with period T_0 corresponds, in the frequency domain, to a discretization (sampling) of the spectrum with step $1 / T_0$ and to a multiplication by a $1 / T_0$ factor. This relation can be seen in Figure 2 along the horizontal arrows that go from $x(t)$ to $\bar{x}(t)$ and from $X(f)$ to $X[k]$.

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