Part I: Fuzzy Logic System

Module 1
Basics of Fuzzy Sets and Fuzzy Relations

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Module 1 Objectives

• To understand basic concepts of fuzzy sets which is the basis of fuzzy logic systems.
• To be able to operate on fuzzy sets using the common fuzzy operations.
• To understand what is fuzzy relations.
• To understand how linguistic variables can be used in fuzzy logic systems.
• To understand how fuzzy rules and a fuzzy knowledge base can be developed.
• To understand how inferencing is done in a fuzzy implication which is in fact fuzzy reasoning.
• At the end of this module the student should be able to do operate on fuzzy sets and fuzzy relations

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1.5 Summary of Module 1
Some Questions!

- What are fuzzy sets?
- Why fuzzy sets?
- What is the difference between a fuzzy set and a crisp set?
- Properties of fuzzy sets?
- Relation of fuzzy sets to the real world?
- How do you operate a fuzzy set on to another fuzzy set → similarly for fuzzy relations?
- How could fuzzy sets be used in a real-world application?

With Fuzzy Logic, Rules Can be Written in a More Natural Way

<table>
<thead>
<tr>
<th>Eg. of Rule-based System without Fuzzy Logic</th>
<th>Eg. of Rule-based System with Fuzzy Logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>• If Temperature is 30°C Then Switch ON Condenser to 80%</td>
<td></td>
</tr>
<tr>
<td>• If Traffic has more than 20 Cars Then Extend Green Light by 20 secs</td>
<td></td>
</tr>
<tr>
<td>• If Temperature is HIGH Then Switch ON Condenser HIGH</td>
<td></td>
</tr>
<tr>
<td>• If Traffic is HEAVY Then Green Should be Extended LONGER</td>
<td></td>
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</tbody>
</table>

1.1 Definition of Fuzzy Sets

1.1.1 Fuzzy Sets compared with Crisp sets

- A ‘crisp’ set, A, can be defined as a set which consists of elements with either full or no membership at all in the set.

- Each item in its universe is either in the set, or not.
Some Facts on Fuzzy Sets

- Fuzzy sets allow the elements in its set to have partial membership.

- Membership from 0 to 1.

- Thus, a fuzzy set is a generalization of an ordinary set by allowing a degree (or grade) of membership for each element.

A fuzzy set uses a common function such as triangular, trapezoidal, gaussian, etc. called membership functions to support the membership value of its elements.

1.1.2 Fuzzy Set Notations

- U is called the universe of discourse and u represents a generic element of U.

- A fuzzy set F in a universe of discourse U is characterized by a membership function,

  \[ \mu_F : U \rightarrow [0, 1] \]

- A fuzzy set F in U is usually represented as a set of ordered pairs of elements u and grade of membership value:

  \[ F = \{(u, \mu_F(u)) | u \in U\} \]
In the above, "+", "and" refer to set union rather than to arithmetic summation, and "/" is simply used to connect an element and its membership value, and has no connection with the arithmetic division.

Fuzzy Sets Properties

- Defined as “a collection of objects all having the same characteristics”
- Notation: U or X, and elements in the universe of discourse are: u or x
- Some examples:
  - Voltages of actuators
  - Speed of cars
  - Error

Example of Universe of Discourse for Persons Height
Example 1.1
In the universe of discourse, the fuzzy set $F$ labeled ‘integer approximately equal to 5’ may be defined as:

$$F = 0.1/2 + 0.4/3 + 0.85/4 + 1.0/5 + 0.85/6 + 0.4/7 + 0.1/8$$

Similarly, the fuzzy subset $F$ labeled ‘integer close to 4’ may be defined as:

$$F = 0.4/2 + 0.8/3 + 1/4 + 0.8/5 + 0.4/6 + 0.1/7 + 0.0/8$$

As discussed, fuzzy set $F$ can be written in the following form:

$$F = \{(2, 0.4), (3, 0.8), (4, 1), (5, 0.8), (6, 0.4), (7, 1), (8, 0)\}$$
• We may use standard functions to represent fuzzy sets.

• The membership functions which are often used in practice include:
  - s-function
  - π-function
  - triangular form
  - trapezoid form
  - exponential form
  - gaussian

### Membership Functions

#### S-membership function

\[
S(u,a,b,c) = \begin{cases} 
0 & \text{for } u < a \\
\frac{(u-a)/(c-a)}{1-2[(u-a)/(c-a)]^2} & \text{for } a \leq u \leq b \\
1 - 2[(u-a)/(c-a)]^2 & \text{for } b \leq u \leq c \\
1 & \text{for } u > c 
\end{cases}
\]

#### II-membership function

\[
x(u,b,c) = \begin{cases} 
S(u,c-b,c-b/2,c) & \text{for } u \leq c \\
1 - S(u,c,c+b/2,c+b) & \text{for } u \geq c
\end{cases}
\]

### Exercise 1.1

1. Write down the mathematical expression for the L-membership function shown.

2. Write down the mathematical expression of this membership function.
Example 1.2

Let the universe of discourse be the interval [0,100] with Y interpreted as age. A fuzzy set F of Y labeled ‘middle age’ may be defined as:

$$F = \{(y, \mu_F(y)) | y \in Y\}$$

What are the implications of the term ‘middle-age’?

A person aged 45 is often considered to be middle-aged. However, what about someone of 38 say? What about other ages? It seems plausible that the required fuzzy set should have a maximum at around 45, and go to zero below 30 and above 60. Usually, a Gaussian-function or T-function gives a good description in such circumstances.

Exercise 1.2

- Suppose a membership function to show “young” is described by the following equation:

  $$m(x) = \begin{cases} 
  1.0 & \text{for } 0 < x \leq 25 \\
  \frac{1}{1 + (\frac{x-25}{5})^2} & \text{for } x > 25 
  \end{cases}$$

- Based on this membership function, write down the membership values of the ages for 10, 20, 25, 30, 35, 40, 50 in this set.

- Draw the graph for this membership function to show “youness”.

1.2 Fuzzy Sets Operations

- 1.2.1 Operations on Classical Sets
- 1.2.2 Basic fuzzy set operations
- 1.2.3 Fuzzy set operations for modifying membership functions
- 1.2.4 Some advanced fuzzy set operations
1.2.1 Operations on Classical Sets

- We can use Venn Diagrams to show operations on classical sets.
- Consider 2 sets: $A$ and $B$ on the universe $X$

Excluded Middle Laws

- Law of the Excluded Middle
  \[ A \cup \overline{A} = X \]
  where $X$ is the universe
- Law of Contradiction
  \[ A \cap \overline{A} = \emptyset \]
  where $\emptyset$ is an empty set or null set
- Draw Venn Diagrams proving these laws?

De Morgan’s Laws

\[ A \cap B = \overline{A} \cup \overline{B} \]
\[ A \cup B = \overline{A} \cap \overline{B} \]

Draw Venn Diagrams showing these laws?
Mapping Sets to Functions

- Set-Theoretic Forms → Function-Theoretic Forms
- f: X → Y can be defined by:
  \[
  \chi_A(x) = \begin{cases} 
  1, & x \in A \\
  0, & x \notin A 
  \end{cases}
  \]

- where \( \chi_A \) expresses membership in set A for the element x in the universe.

Operations for Function-theoretic terms

Consider 2 sets: A and B, on the universe Y:

- Union:
  \[ A \cup B \rightarrow \chi_{A \cup B}(x) = \chi_A \lor \chi_B = \max(\chi_A, \chi_B) \]

- Intersection:
  \[ A \cap B \rightarrow \chi_{A \cap B}(x) = \chi_A \land \chi_B = \min(\chi_A, \chi_B) \]

- Complement:
  \[ A \rightarrow \chi_A^c(x) = 1 - \chi_A(x) \]

1.2.2 Basic Fuzzy Set Operations

- The use of fuzzy sets provides a basis for the systematic manipulation of vague and imprecise concepts using fuzzy set operations performed by manipulating the membership functions.

- Let A and B be two point-valued fuzzy sets in U with membership functions \( \mu_A \) and \( \mu_B \), respectively as given in the next page.

Consider these two fuzzy sets:

How do you perform an operation of one fuzzy set to another?
**Equality**

Two fuzzy sets $A$ and $B$ are equal if they are defined on the same universe and the membership function is the same for both, that is:

$$A = B \iff \mu_A(u) = \mu_B(u) \quad \forall u \in U$$

**Union**

The union of two fuzzy sets $A$ and $B$ is the fuzzy set whose membership function is given by:

$$\mu_{A \cup B}(u) = \max\{\mu_A(u), \mu_B(u)\} \quad \forall u \in U$$

**Subset**

$$A \subseteq B \iff \mu_A(u) \leq \mu_B(u) \quad \forall u \in U$$

Draw the graph showing “subset”

**Complement**

The complement of a (normalized) fuzzy set $A$ with membership function is defined as the fuzzy set on the same universe with membership function:

$$\mu'_A(u) = 1 - \mu_A(u) \quad \forall u \in U$$

**Intersection**

The intersection of two fuzzy sets $A$ and $B$ is the fuzzy set whose membership function is given by:

$$\mu_{A \cap B}(u) = \min\{\mu_A(u), \mu_B(u)\} \quad \forall u \in U$$

1.2.3 Fuzzy Set Operations for Modifying Membership Functions

- The shape of membership functions of fuzzy sets can be changed.
- This can be done by using specific operations.

**Power**

- If $p$ is a positive number and $A$ is a fuzzy set with membership function $\mu_A(x)$, then $A$ on power $p$, denoted by $A^p$, is defined as:

$$A^p = \{(x, \mu_A(x))^p\} = \{(x, (\mu_A(x))^p)\}$$
**Concentration**

- A fuzzy set $A$ can be 'concentrated' by modifying its membership function so as to accentuate the membership of the higher membership elements.

$$\mu_{CON(A)}(u) = (\mu_A(u))^2 \quad \forall u \in U$$

**Dilation**

- A fuzzy set $A$ can be 'dilated' by modifying its membership function to increase the importance of lower membership elements.

$$\mu_{DIL(A)}(u) = (\mu_A(u))^3 \quad \forall u \in U$$

**Intensification**

- This operation moves the normalized fuzzy set closer to being crisp, by enhancing the membership value of those elements whose membership was above 0.5 and diminishing that of those elements with membership below 0.5.

$$\mu_{INT(A)}(u) = \begin{cases} 2(\mu_A(u))^2 & \text{for } 0 \leq \mu_A(u) \leq 0.5 \\ 1 - 2(1 - \mu_A(u))^2 & \text{for } 0.5 \leq \mu_A(u) \leq 1 \end{cases}$$

**Exercise 1.3 a**

- Consider the fuzzy set $A$:

$$A = \{(1,0.1), (2,0.8), (3,1), (4,0.2), (5,0.5)\}$$

- Solve:
  
  - $oA^3$
  - $oCONN(A)$
  - $oDIL(A)$
  - $oINT(A)$

- Draw graphs showing the operations
Exercise 1.3
b

- For the fuzzy set with the following membership function (take the elements of the universe of discourse to be 0, 0.5, 1, 1.5 and 2), find and draw the graphs of
  - CON(A)
  - DIL(A)

\[ \mu_A(u) = \begin{cases} 
  u & \text{for } 0 \leq u \leq 1 \\
  2 - u & \text{for } 1 \leq u \leq 2 
\end{cases} \]

Exercise 1.4

Consider the fuzzy sets A and B where:

A = \{(x_1, 0.2), (x_2, 0.4), (x_3, 1), (x_4, 0.3), (x_5, 0.1)\}

B = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.6), (x_4, 0.4)\}

Solve:
- algebraic product
- algebraic sum.

1.2.3 Some Advanced Fuzzy Set Operations

- **Algebraic product**
  The algebraic product of two fuzzy sets A and B is the multiplication of the membership functions which is given by:

\[ \mu_{A \times B}(u) = \mu_A(u) \cdot \mu_B(u) \quad \forall u \in U \]

- **Algebraic sum**
  The algebraic sum of two fuzzy sets A and B is also known as probabilistic sum and is given by the following expression:

\[ \mu_{A + B}(u) = \mu_A(u) + \mu_B(u) - \mu_A(u) \cdot \mu_B(u) \quad \forall u \in U \]

- **Bounded product**
  The bounded product of two fuzzy sets A and B with membership functions is the fuzzy set whose membership function is given by:

\[ \mu_{A \cdot B}(u) = \max(0, \mu_A(u) + \mu_B(u) - 1) \quad \forall u \in U \]

- **Bounded sum**
  The bounded sum of two fuzzy sets A and B with membership functions and is the fuzzy set whose membership function is given by:

\[ \mu_{A + B}(u) = \min(1, \mu_A(u) + \mu_B(u)) \quad \forall u \in U \]

where ‘\(+\)’ is the arithmetic sum operator.
Drastic product

The drastic product of two fuzzy sets A and B with membership function and is the fuzzy set whose membership function is given by:

\[ \mu_{A \times B}(u) = \begin{cases} \mu_A(u) & \text{for } \mu_B(u) = 1 \\ \mu_B(u) & \text{for } \mu_A(u) = 1 \\ 0 & \text{for } \mu_A(u), \mu_B(u) < 1 \end{cases} \]

Exercise 1.5:

Consider the fuzzy sets A and B where:

A\!=(x_1, 0.2), (x_2, 0.3), (x_3, 1), (x_4, 0.5), (x_5, 0.1), \}
B\!=(x_1, 0.0), (x_2, 0.3), (x_3, 0.6), (x_4, 0.7), (x_5, 0.9)\}

Perform the following on A and B:

1. Algebraic product
2. Algebraic sum
3. Bounded product
4. Bounded sum
5. Drastic product