Analysis and Transmissions of Signals:

Electrical engineers instinctively think of signals in terms of their frequency spectra and think of systems in terms of their frequency responses. Even teenagers know about audio signals having a bandwidth of 20 kHz and good-quality loud speakers responding up to 20 kHz. This is basically thinking in the frequency domain. In the previous lectures, we discussed spectral representation of periodic signals (Fourier series). In this lecture, we extend this spectral representation to aperiodic signals.

2.1 Aperiodic Signal Representation by Fourier Integral:

Applying a limiting process, we now show that an aperiodic signal can be expressed as a continuous sum (integral) of everlasting exponentials. To represent an aperiodic signal \( g(t) \), such as the one shown in Figure (2.1a) by everlasting exponential signals, let us construct a new periodic signal \( g_{T_0}(t) \) formed by repeating the signal \( g(t) \) every \( T_0 \) seconds, as shown in figure (2.1b).

The period \( T_0 \) is made long enough to avoid overlap between the repeating pulses. The periodic signal \( g_{T_0}(t) \) can be represented by an exponential Fourier series. If we let \( T_0 \to \infty \), the pulses in the periodic signal repeat after an infinite interval, and therefore

\[
\lim_{T_0 \to \infty} g_{T_0}(t) = g(t)
\]

Thus, the Fourier series representing \( g_{T_0}(t) \) will also represent \( g(t) \) in the limit \( T_0 \to \infty \). The exponential Fourier series for \( g_{T_0}(t) \) is given by

\[
g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 t}
\]

2.1

In which
\[ D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-j n \omega_0 t} dt \]  \hspace{1cm} 2.2a

And

\[ \omega_0 = \frac{2\pi}{T_0} \]  \hspace{1cm} 2.2b

Observe that integrating \( g_{T_0}(t) \) over \((-T_0/2, T_0/2)\) is the same as integrating \( g(t) \) over \((-\infty, \infty)\). Therefore, equation (2.2a) can be expressed as

\[ D_n = \frac{1}{T_0} \int_{-\infty}^{\infty} g(t) e^{-j n \omega_0 t} dt \]  \hspace{1cm} 2.2c

It is interesting to see how the nature of the spectrum changes as \( T_0 \) increases. To understand this fascinating behavior, let us define \( G(\omega) \), a continuous function of \( \omega \), as

\[ G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j \omega t} dt \]  \hspace{1cm} 2.3

A glance at equations (2.2c) and (2.3) shows that

\[ D_n = \frac{1}{T_0} G(n \omega_0) \]  \hspace{1cm} 2.4

This shows that the Fourier coefficients \( D_n \) are \((1/T_0 \text{ times})\) the samples of \( G(\omega) \) uniformly spaced at intervals of \( \omega_0 \) rad/s, as shown in Figure (2.2a).

Therefore, \( 1/T_0 G(\omega) \) is the envelope for the coefficients \( D_n \). We now let \( T_0 \to \infty \) by doubling \( T_0 \) repeatedly. Doubling \( T_0 \) halves the fundamental frequency \( \omega_0 \), so that there are now twice as many components (samples) in the spectrum. However, by doubling \( T_0 \), the envelop \( 1/T_0 G(\omega) \) is halved, as shown in Figure (2.2b). If we continue this process of doubling \( T_0 \) repeatedly, the spectrum progressively becomes denser while its
magnitude becomes smaller. Note, however, that the relative shape of the envelope remains the same. In the limit as $T_0 \to \infty$, $\omega_0 \to 0$ and $D_n \to 0$. This means that the spectrum is so dense that the spectral components are spaced at zero (infinitesimal) intervals. At the same time, the amplitude of each component is zero (infinitesimal). This sounds like Alice in Wonderland, but as we shall see, these are the classic characteristics of a very familiar phenomenon.

Substitution of equation (2.4) in equation (2.1) yields

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{G(n\omega_0)}{T_0} e^{jn\omega_0t}$$

As $T_0 \to \infty$, $\omega_0$ becomes infinitesimal ($\omega_0 \to 0$). Because of this, we shall replace $\omega_0$ by a more appropriate notation, $\Delta \omega$. In terms of this new notation, equation (2.2b) becomes

$$\Delta \omega = \frac{2\pi}{T_0}$$

Moreover, equation (2.5) becomes

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} \left[ \frac{G(n\Delta \omega)}{2\pi} \right] e^{jn\Delta \omega t}$$

Equation (2.6a) shows that $g_{T_0}(t)$ can be expressed as a sum of everlasting exponentials of frequencies $0, \pm \Delta \omega, \pm 2\Delta \omega, \pm 3\Delta \omega, \ldots$ (the Fourier series). The amount of the component of frequency $n\Delta \omega$ is $[G(n\Delta \omega)\Delta \omega]/2\pi$. In the limit as $T_0 \to \infty$, $\Delta \omega \to 0$ and $g_{T_0}(t) \to g(t)$. Therefore,

$$g(t) = \lim_{T_0 \to \infty} g_{T_0}(t) = \lim_{\Delta \omega \to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} G(n\Delta \omega) e^{jn\Delta \omega t} \Delta \omega$$

The sum on the right-hand side of equation (2.6b) can be viewed as the area under the function $G(\omega)e^{j\omega t}$, as shown in Figure (2.3). Therefore,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)e^{j\omega t} d\omega$$
The integral on the right-hand side is called the **Fourier integral**. We have now succeeded in representing an aperiodic signal \( g(t) \) by a Fourier integral.

This integral is a Fourier series (in the limit) with fundamental frequency \( \Delta \omega \to 0 \), as seen from equation (2.6). The amount of the exponential \( e^{jn\Delta \omega t} \) is \( \frac{1}{2\pi} \). Thus, the function \( G(\omega) \) given by equation (2.3) acts as a spectral function.

We call \( G(\omega) \) the **direct** Fourier transform of \( g(t) \), and \( g(t) \) the **inverse** Fourier transform of \( G(\omega) \). The statement conveys the same information that \( g(t) \) and \( G(\omega) \) is a Fourier transform pair. Symbolically, this expressed as

\[
G(\omega) = F[g(t)] \quad \text{and} \quad g(t) = F^{-1}[G(\omega)]
\]

Or

\[
g(t) \leftrightarrow G(\omega)
\]

To recapitulate,

\[
G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \tag{2.8a}
\]

And

\[
g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega \tag{2.8b}
\]

It is helpful to keep in mind that the Fourier integral in Equation (2.8b) is of the nature of a Fourier series with fundamental frequency \( \Delta \omega \) approaching zero [equation (2.6b)]. Therefore, most of the discussion and properties of Fourier series apply to the Fourier transform as well. We can plot the spectrum \( G(\omega) \) as a function of \( \omega \). Since \( G(\omega) \) is complex, we have both amplitude and angle (or phase) spectra:
\[ G(\omega) = |G(\omega)|e^{j\theta_g(\omega)} \]

In which \(|G(\omega)|\) is the amplitude and \(\theta_g(\omega)\) is the angle (or phase) of \(G(\omega)\). From equation (2.8a),

\[ G(-\omega) = \int_{-\infty}^{\infty} g(t)e^{j\omega t} dt \]

**Conjugate Symmetry Property**

From this equation and equation (2.8a), it follows that if \(g(t)\) is a real function of \(t\), then \(G(\omega)\) and \(G(-\omega)\) are complex conjugate, that is,

\[ G(-\omega) = G^*(\omega) \]

2.9

Therefore,

\[ |G(-\omega)| = |G(\omega)| \]

2.10a

\[ \theta_g(-\omega) = -\theta_g(\omega) \]

2.10b

Thus, for real \(g(t)\), the amplitude spectrum \(|G(\omega)|\) is an even function and the phase spectrum \(\theta_g(\omega)\) is an odd function of \(\omega\). This property (the **conjugate symmetry property**) is valid only for real \(g(t)\). These results were derived earlier for the Fourier spectrum of a periodic signal and should come as no surprise. *The transform \(G(\omega)\) is frequency-domain specification of \(g(t)\).*

**Example 2.1:** Find the Fourier transform of \(e^{-at}u(t)\).

![Figure 2.4: \(e^{-at}u(t)\) and its Fourier spectra.](image)

**Solution:** By definition [equation (2.8a)],
\[
G(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \frac{-1}{a + j\omega} e^{-(a+j\omega)t} \bigg|_{0}^{\infty}
\]

But \(|e^{-j\omega}| = 1\). Therefore, as \(t \to \infty\), \(e^{-(a+j\omega)t} e^{-j\omega t} = e^{-at} e^{-j\omega t} = 0\) if \(a > 0\). Therefore,

\[
G(\omega) = \frac{1}{a + j\omega} \quad a > 0
\]

Expressing \(a + j\omega\) in the polar form as \(\sqrt{a^2 + \omega^2} e^{j\tan^{-1}(\frac{\omega}{a})}\), we obtain

\[
G(\omega) = \frac{1}{\sqrt{a^2 + \omega^2}} e^{j\tan^{-1}(\frac{\omega}{a})}
\]

Therefore,

\[
|G(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} \quad \text{and} \quad \theta_g(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)
\]

The amplitude spectrum \(|G(\omega)|\) and the phase spectrum \(\theta_g(\omega)\) are shown in figure (2.4b). Observe that \(|G(\omega)|\) is an even function of \(\omega\), and \(\theta_g(\omega)\) is an odd function of \(\omega\), as expected.

**Linearity of the Fourier Transform**

The Fourier transform is linear; that is, if

\[
g_1(t) \leftrightarrow G_1(\omega) \quad \text{and} \quad g_2(t) \leftrightarrow G_2(\omega)
\]

Then

\[
a_1g_1(t) + a_2g_2(t) \leftrightarrow a_1G_1(\omega) + a_2G_2(\omega)
\]

### 2.2 Transformation of Some Useful Functions:

For convenience, we now introduce a compact notation for some useful functions such as **gate, triangle, and interpolation** functions.

**Unit Gate Function**: We define a unit gate function \(\text{rect}(x)\) as a gate pulse of unit height and unit width, centered at the origin, as shown in Figure (2.7a):
The gate pulse in Figure (2.7b) is the unit gate pulse $\text{rect}(x)$ expanded by a factor $\tau$ and therefore can be expressed as $\text{rect}(x/\tau)$. Observe that $\tau$, the denominator of the argument of $\text{rect}(x/\tau)$, indicates the width of the pulse.

**Unit Triangle Function:** We define a unit triangle function $\Delta(x)$ as a triangular pulse of unit height and unit width, centered at the origin, as shown in figure (2.8a):

\[
\Delta(x) = \begin{cases} 
0 & |x| > \frac{1}{2} \\
\frac{1}{2} & |x| = \frac{1}{2} \\
1 & |x| < \frac{1}{2} 
\end{cases}  
\]  
2.15

The pulse in Figure (2.8a) is $\Delta(x/\tau)$. Observe that here, as for the gate pulse, the denominator $\tau$ of the argument of $\Delta(x/\tau)$ indicates the pulse width.
**Interpolation Function sinc(x):** The function \(\frac{\sin(x)}{x}\) is the “sine over argument” function denoted by \(sinc(x)\). This function

\[
sinc(x) = \frac{\sin \pi x}{\pi x}
\]

Plays an important role in signal processing. It is also known as the **filtering** or **interpolation function**. We define

\[
sinc(x) = \frac{\sin x}{x}
\]

Inspection of equation (2.16) shows that

1. \(sinc(x)\) is an even function of \(x\).
2. \(sinc(x) = 0\) When \(\sin x = 0\) except at \(x = 0\), where it is indeterminate. This means that \(sinc(x) = 0\) for \(x = \pm \pi, \pm 2\pi, \pm 3\pi, \ldots\).
3. Using L'Hopital’s rule, we find \(sinc(0) = 1\).
4. \(sinc(x)\) is the product of an oscillating signal \(\sin x\) (of period \(2\pi\)) and a monotonically decreasing function \(1/x\). Therefore, \(sinc(x)\) exhibits sinusoidal oscillations of period \(2\pi\), with amplitude decreasing continuously as \(1/x\).

Figure (2.9a) shows \(sinc(x)\). Observe that \(sinc(x) = 0\) for values of \(x\) that are positive and negative integral multiples of \(\pi\). Figure (2.9b) shows \(sinc(3\omega/7)\). The argument \(3\omega/7 = \pi\) when \(\omega = 7\pi/3\). Therefore, the first zero of this function occurs at \(\omega = 7\pi/3\).
**Example 2.2:** Find the Fourier transform of $g(t) = \text{rect}(t/\tau)$ [Figure (2.10a)].

![Figure (2.9): Sinc pulse.](image)

![Figure (2.10): Gate pulse and its Fourier spectrum.](image)

**Solution:** We have

$$G(\omega) = \int_{-\infty}^{\infty} \text{rect} \left( \frac{t}{\tau} \right) e^{-j\omega t} dt$$

Since $\text{rect} \left( \frac{t}{\tau} \right) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$G(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} (e^{-j\omega \tau/2} - e^{j\omega \tau/2}) = \frac{2 \sin(\omega \tau/2)}{\omega}$$
\[
\frac{\tau \sin(\omega t/2)}{(\omega t/2)} = \tau \operatorname{sinc}
\left(\frac{\omega t}{2}\right)
\]

Therefore,
\[
\text{rect}\left(\frac{t}{\tau}\right) \leftrightarrow \tau \operatorname{sinc}\left(\frac{\omega t}{2}\right)
\]

Recall that \(\operatorname{sinc}(x) = 0\) when \(x = \pm n\pi\). Hence, \(\operatorname{sinc}\left(\frac{\omega t}{2}\right) = 0\) when \(\frac{\omega t}{2} = \pm n\pi\); that is, when \(\omega = \pm 2n\pi/\tau\) \((n = 1, 2, 3, \ldots)\) as shown in Figure (2.10b). Observe that in this case \(G(\omega)\) happens to be real. Hence, we may convey the spectral information by a single plot of \(G(\omega)\) shown in Figure (2.10b).

**Bandwidth of \(\text{rect}\left(\frac{t}{\tau}\right)\)**

The spectrum \(G(\omega)\) in Figure (2.10) peaks at \(\omega = 0\) and decays at higher frequencies. Therefore, \(\text{rect}(t/\tau)\) is a low-pass signal with most of the signal energy in lower frequency components. **Signal bandwidth** is the difference between the highest (significant) frequency and the lowest (significant) frequency in the signal spectrum. Strictly speaking, because the spectrum extends from zero to \(\infty\), the bandwidth is \(\infty\) in the present case. However, much of the spectrum extends from zero to \(\infty\); the bandwidth is \(\infty\) in the present case. However, much of the spectrum is concentrated within the first lobe (from \(\omega = 0\) to \(\omega = 2\pi/\tau\)), and we may consider \(\omega = 2\pi/\tau\) being highest (significant) frequency in the spectrum. Therefore, a rough estimate of the bandwidth of a rectangular pulse of width \(\tau\) seconds is \(2\pi/\tau\) rad/sec, or \(1/\tau\) Hz. Note the reciprocal relationship of the pulse with its bandwidth. We shall observe later that this result is true in general.

**Example 2.3:** Find the Fourier transform of the unit impulse \(\delta(t)\).

**Solution:** Using the sampling property of impulse, we obtain

\[
F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} \, dt = 1
\]

Or

\[
\delta(t) \leftrightarrow 1
\]

Figure (2.11) shows \(\delta(t)\) and its spectrum.
**Example 2.4:** Find the inverse Fourier transform of $\delta(\omega)$.

**Solution:** From equation (2.8b) and the sampling property of the impulse function,

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$$

Therefore,

$$\frac{1}{2\pi} \leftrightarrow \delta(\omega) \quad 2.19a$$

Or

$$1 \leftrightarrow 2\pi\delta(\omega) \quad 2.19b$$

This shows that the spectrum of a constant signal $g(t) = 1$ is an impulse $2\pi\delta(\omega)$, as shown in Figure (2.12).

The result [equation (2.19b)] also could have been anticipated on qualitative grounds. Recall that the Fourier transform of $g(t)$ is a spectral representation of $g(t)$ in terms of everlasting exponential components of the form $e^{j\omega t}$. Now to represent a constant signal $g(t) = 1$, we need a single everlasting exponential $e^{j\omega t}$ with $\omega = 0$. This result in a spectrum at a single frequency $\omega = 0$. Another way of looking at the situation is that $g(t) = 1$ is a dc signal, which has a single frequency $\omega = 0$ (dc).
Example 2.5: Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.

Solution: Using the sampling property of the impulse function, we obtain

$$F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Therefore,

$$\frac{1}{2\pi} e^{j\omega_0 t} \leftrightarrow \delta(\omega - \omega_0)$$

Or

$$e^{j\omega_0 t} \leftrightarrow 2\pi \delta(\omega - \omega_0) \quad 2.20b$$

This result shows that the spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse at $\omega = \omega_0$. We reach the same conclusion by qualitative reasoning. To represent the everlasting exponential $e^{j\omega_0 t}$. We need a single everlasting exponential $e^{j\omega_0 t}$ with $\omega = \omega_0$. Therefore, the spectrum consists of a single component at frequency $\omega = \omega_0$.

From equation (2.20a), it follows that

$$e^{-j\omega_0 t} \leftrightarrow 2\pi \delta(\omega + \omega_0) \quad 2.20b$$

Example 2.6: Find the Fourier transforms of the everlasting sinusoid $\cos \omega_0 t$.

Solution: Recall the Euler formula

$$\cos \omega_0 t = \frac{1}{2} \left( e^{j\omega_0 t} + e^{-j\omega_0 t} \right)$$

Adding equations (2.20a) and (2.20b), and using the above formula, we obtain

$$\cos \omega_0 t \leftrightarrow \pi \left[ \delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right] \quad 2.21$$

The spectrum of $\cos \omega_0 t$ consists of two impulses at $\omega_0 \text{ and } -\omega_0$, as shown in Figure (2.13). The result also follows from qualitative reasoning. An everlasting sinusoid $\cos \omega_0 t$...
can be synthesized by two everlasting exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. Therefore, the Fourier spectrum consists of only two components of frequenies $\omega_0$ and $-\omega_0$.

Figure (2.13): Cosine signal and its Fourier spectrum.

**Example 2.7:** Find the Fourier transfoem of the sign Function $\text{sgn} \ t$ (pronounced signum $t$), shown in Figure (2.14). its value is +1 or -1, depending on whether $t$ is positive or negative:

$$\text{sgn} \ t = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$  \hspace{1cm}  \text{(2.22)}$$

Solution: The transform of $\text{sgn} \ t$ can be obtained by considering $\text{sgn} \ t$ as a sum of two exponentials, as shown in Figure (2.14), in the limit as $a \to 0$:

$$\text{sgn} \ t = \lim_{a \to 0} [e^{-at}u(t) - e^{at}u(-t)]$$

Therefore,

$$F[\text{sgn} \ t] = \lim_{a \to 0} \{F[e^{-at}u(t)] - F[e^{at}u(-t)]\}$$

$$= \lim_{a \to 0} \left( \frac{1}{a + j\omega} - \frac{1}{a - j\omega} \right)$$

$$= \lim_{a \to 0} \left( \frac{-2j\omega}{a^2 + \omega^2} \right) = \frac{2}{j\omega}$$  \hspace{1cm}  \text{(2.23)}$$

Table 2.1 shows the commone and mostly used functions with their Fourier transforms.

**Table 2.1: Short Table for Fourier Transforms**
<table>
<thead>
<tr>
<th>#</th>
<th>$g(t)$</th>
<th>$G(\omega)$</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e^{-at}u(t)$</td>
<td>$\frac{1}{a + j\omega}$</td>
<td>$a &gt; 0$</td>
</tr>
<tr>
<td>2</td>
<td>$e^{at}u(-t)$</td>
<td>$\frac{1}{a - j\omega}$</td>
<td>$a &gt; 0$</td>
</tr>
<tr>
<td>3</td>
<td>$e^{-at}u(t)$</td>
<td>$\frac{2a}{a^2 + \omega^2}$</td>
<td>$a &gt; 0$</td>
</tr>
<tr>
<td>4</td>
<td>$te^{-at}u(t)$</td>
<td>$\frac{1}{(a + j\omega)^2}$</td>
<td>$a &gt; 0$</td>
</tr>
<tr>
<td>5</td>
<td>$tn e^{-at}u(t)$</td>
<td>$\frac{n!}{(a + j\omega)^{n+1}}$</td>
<td>$a &gt; 0$</td>
</tr>
<tr>
<td>6</td>
<td>$\delta(t)$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>$2\pi\delta(\omega)$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$e^{j\omega_0 t}$</td>
<td>$2\pi\delta(\omega - \omega_0)$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$\cos(\omega_0 t)$</td>
<td>$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$\sin(\omega_0 t)$</td>
<td>$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$u(t)$</td>
<td>$\pi\delta(\omega) + \frac{1}{j\omega}$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$\text{sgn } t$</td>
<td>$\frac{2}{j\omega}$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>$\cos(\omega_0 t) u(t)$</td>
<td>$\frac{\pi}{2} \left[ \delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right] + \frac{j\omega}{\omega_0^2 - \omega^2}$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$\sin(\omega_0 t) u(t)$</td>
<td>$\frac{\pi}{2j} \left[ \delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right] + \frac{\omega_0}{\omega_0^2 - \omega^2}$</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>$e^{-at} \sin(\omega_0 t) u(t)$</td>
<td>$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$</td>
<td>$a &gt; 0$</td>
</tr>
<tr>
<td>16</td>
<td>$e^{-at} \cos(\omega_0 t) u(t)$</td>
<td>$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$</td>
<td>$a &gt; 0$</td>
</tr>
<tr>
<td>17</td>
<td>$\text{rect} \left( \frac{t}{\tau} \right)$</td>
<td>$\tau \text{sinc} \frac{\omega \tau}{2}$</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>$\frac{W}{\pi} \text{sinc}(Wt)$</td>
<td>$\text{rect} \left( \frac{\omega}{2W} \right)$</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>$\Delta \left( \frac{t}{\tau} \right)$</td>
<td>$\frac{\tau}{2} \text{sinc}^2 \left( \frac{\omega \tau}{4} \right)$</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>$\sum_{n=-\infty}^{\infty} \delta(t - nT')$</td>
<td>$\omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$</td>
<td>$\omega_0 = \frac{2\pi}{T}$</td>
</tr>
<tr>
<td>21</td>
<td>$e^{-t^2/2\sigma^2}$</td>
<td>$\sigma \sqrt{2\pi} e^{-\sigma^2 \omega^2/2}$</td>
<td></td>
</tr>
</tbody>
</table>

2.3 Some Properties of the Fourier Transform:
We now study some of the important properties of the Fourier transform and their implications as well as their applications.

### 2.3.1 Symmetry of Direct and Inverse Transform Operations - Time-Frequency Duality

Equations (2.8) show an interesting fact: the direct and the inverse transform operations are remarkably similar. These operations, required to go from \( g(t) \) to \( G(\omega) \) and then from \( G(\omega) \) to \( g(t) \). There are only two minor differences in these operations: the factor \( 2\pi \) appears only in the inverse operator, and the exponential indices in the two operations have opposite signs. Otherwise the two operations are symmetrical. This is called the duality of time and frequency.

The duality principle may be compared with a photograph and its negative. A photograph can be obtained from its negative, and by using an identical procedure, the negative can be obtained from the photograph.

For any result or relationship between \( g(t) \) and \( G(\omega) \), there exists a dual result or relationship, obtained by interchanging the roles of \( g(t) \) and \( G(\omega) \) in the original result (along with some minor modifications arising because of the factor \( 2\pi \) and a sign change).

For example, the time-shifting property (to be discussed later), states that if \( g(t) \leftrightarrow G(\omega) \) then

\[
g(t - t_0) \leftrightarrow G(\omega)e^{-j\omega t_0}
\]

The dual of this property (the frequency-shifting property) states that

\[
g(t)e^{j\omega_0 t} \leftrightarrow G(\omega - \omega_0)
\]

### 2.3.2 Symmetry Property

This property states that if

\[
g(t) \leftrightarrow G(\omega)
\]

Then
Example 2.8: In this example we shall apply the symmetry property [Equation (2.24)] to the pair in Figure (2.16a).

\[ G(t) \leftrightarrow 2\pi g(-\omega) \] 

**Proof:** From Equation (2.8b),

\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x)e^{jtx}dx \]

Hence,

\[ 2\pi g(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x)e^{-jtx}dx \]

Changing \( t \) to \( \omega \) yields equation (2.24).

Solution: Form equation (2.17) we have

\[ \text{rect} \left( \frac{t}{T} \right) \leftrightarrow T \text{sinc} \left( \frac{\omega T}{2} \right) \]

Also \( G(t) \) is the same as \( G(\omega) \) with \( \omega \) replaced by \( t \), and \( g(-\omega) \) is the same as \( g(t) \) with \( t \) replace by \( -\omega \). Therefore, the symmetry property (2.24) yields
In equation (2.26) we used the fact that \( \text{rect}(−x) = \text{rect}(x) \) because \( \text{rect} \) is an even function. Figure (2.16b) shows this pair graphically. Observe the interchange of the roles of \( t \) and \( \omega \) (with the minor adjustment of the factor \( 2\pi \)). This result appears as pair 18 in table (2.1) with \( \tau/2 = W \).

### 2.3.3 Scaling Property:

If

\[
g(t) \leftrightarrow G(\omega)
\]

Then, for any real constant \( a \),

\[
g(at) \leftrightarrow \frac{1}{|a|}G\left(\frac{\omega}{a}\right)
\]

The function \( g(at) \) represents the function \( g(t) \) compressed in time by a factor \( a \). Similarly, a function \( G(\omega/a) \) represents the function \( G(\omega) \) expanded in frequency by the same factor \( a \). The scaling property states that time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression.

**Example 2.9:** Show that

\[
g(−t) \leftrightarrow G(−\omega)
\]

Using this result and the fact that \( e^{-at}u(t) \leftrightarrow 1/(a+j\omega) \), find the Fourier transform of \( e^{at}u(−t) \) and \( e^{-a|t|} \).

**Solution:** Equation (2.28) follows from equation (2.27) by letting \( a = -1 \). Application of equation (2.28) to pair 1 of table (2.1) yeilds

\[
e^{at}u(−t) \leftrightarrow \frac{1}{a−j\omega}
\]

Also

\[
e^{-a|t|} = e^{at}u(t) + e^{at}u(−t)
\]

Therefore,

\[
e^{-a|t|} \leftrightarrow \frac{1}{a+j\omega} + \frac{1}{a−j\omega} = \frac{2a}{a^2 + \omega^2}
\]
2.3.4 Time - Shifting Property:

If

\[ g(t) \leftrightarrow G(\omega) \]

Then

\[ g(t - t_0) \leftrightarrow G(\omega)e^{-j\omega t_0} \]  

Example 2.10: Find the Fourier transform of \( e^{-a|t-t_0|} \).

Solution: This function is a time-shifted version of \( e^{-a|t|} \). From equations (2.39) and (2.30) we have

\[ e^{-a|t-t_0|} \leftrightarrow \frac{2a}{a^2 + \omega^2} e^{-j\omega t_0} \]  

The spectrum of \( e^{-a|t-t_0|} \) is the same as that of \( e^{-a|t|} \), except for an odd phase shift of \(-\omega t_0\). Note that the time delay \( t_0 \) causes a linear phase spectrum \(-\omega t_0\). See Figure (2.20)

\[ g(t) = e^{-a|t-t_0|} \]

\[ |G(\omega)| = \frac{2a}{a^2 + \omega^2} \]

\[ \theta_\omega(\omega) = -\omega t_0 \]

|Figure (2.20): Effect of time shifting on the Fourier spectrum of a signal.|

2.3.5 Frequency - Shifting Property: (important)

If

\[ g(t) \leftrightarrow G(\omega) \]

Then

\[ g(t)e^{j\omega_0 t} \leftrightarrow G(\omega - \omega_0) \]  

This property states that multiplication of a signal by a factor \( e^{j\omega_0 t} \) shifts the spectrum of that signal by \( \omega = \omega_0 \). Note the duality between the time-shifting and the frequency-shifting properties.

Now, changing \( \omega_0 \) to \(-\omega_0\) in equation (2.33) yields
Because $e^{j\omega_0 t}$ is not a real function that can be generated, frequency shifting in practice is achieved by multiplying $g(t)$ by a sinusoid. This can be seen from the fact that

$$g(t) \cos \omega_0 t = \frac{1}{2} [g(t) e^{j\omega_0 t} + g(t) e^{-j\omega_0 t}]$$

From equations (2.33) and (2.34), it follows that

$$g(t) \cos \omega_0 t \leftrightarrow \frac{1}{2} [G(\omega - \omega_0) + G(\omega + \omega_0)]$$  \hspace{1cm} 2.35$$

This shows that the multiplication of a signal $g(t)$ by a sinusoid of frequency $\omega_0$ shifts the spectrum $G(\omega)$ by $\pm \omega_0$. Multiplication of a sinusoid $\cos \omega_0 t$ by $g(t)$ amounts to modulating the sinusoid amplitude. This type of modulation is known as **amplitude modulation**. The sinusoid $\cos \omega_0 t$ is called the **carrier**, the signal $g(t)$ is the **modulating signal**.

To sketch a signal $g(t) \cos \omega_0 t$, we observe that

$$g(t) \cos \omega_0 t = \begin{cases} g(t) & \text{when } \cos \omega_0 t = 1 \\ -g(t) & \text{when } \cos \omega_0 t = -1 \end{cases}$$

Therefore, $g(t) \cos \omega_0 t$ touches $g(t)$ when the sinusoid $\cos \omega_0 t$ is at its positive peaks and touches $-g(t)$ when $\cos \omega_0 t$ is at its negative peaks. This means that $g(t)$ and $-g(t)$ act as envelopes for the signal $g(t) \cos \omega_0 t$ [see Figure (2.21c)]. The signal $-g(t)$ is a mirror image of $g(t)$ about the horizontal axis. Figure (2.21) shows the signals $g(t)$, $g(t) \cos \omega_0 t$ and their spectra.

**Shifting the Phase Spectrum of a Modulated Signal:**

We can shift the phase of each spectral component of a modulated signal by a constant amount $\theta_0$ merely by using a carrier $\cos(\omega_0 t + \theta_0)$ instead of $\cos(\omega_0 t)$. If a signal $g(t)$ is multiplied by $\cos(\omega_0 t + \theta_0)$, then using an argument similar to that used to derive Equation (2.35), we can show that

$$g(t) \cos(\omega_0 t + \theta_0) \leftrightarrow \frac{1}{2} [G(\omega - \omega_0)e^{j\theta_0} + G(\omega + \omega_0)e^{-j\theta_0}]$$  \hspace{1cm} 2.36$$

For a special case when $\theta_0 = -\pi/2$, equation (2.36) becomes

$$g(t) \sin(\omega_0 t) \leftrightarrow \frac{1}{2} [G(\omega - \omega_0)e^{-j\pi/2} + G(\omega + \omega_0)e^{j\pi/2}]$$  \hspace{1cm} 2.37$$
Example 2.12: Find and sketch the Fourier transform of the modulated signal $g(t) \cos \omega_0 t$ in which $g(t)$ is a gate pulse $\text{rect}(t/T)$, as shown in Figure (2.22a).
**Solution:** The pulse $g(t)$ is the same rectangular pulse shown in Figure (2.10a) (with $\tau = T$). From pair 17 of Table (2.1), we find $G(\omega)$, the Fourier transform of $g(t)$, as

$$\text{rect} \left( \frac{t}{T} \right) \leftrightarrow T \text{sinc} \left( \frac{\omega T}{2} \right)$$

This spectrum $G(\omega)$ is shown in Figure (2.22b). The signal $g(t) \cos \omega_0 t$ is shown in Figure (2.22c). Form equation (2.35) it follows that

$$g(t) \cos \omega_0 t \leftrightarrow \frac{1}{2} [G(\omega + \omega_0) + G(\omega - \omega_0)]$$

This spectrum of $g(t) \cos \omega_0 t$ is obtained by shifting $G(\omega)$ in Figure (2.22b) to the left by $\omega_0$ and also to the right by $\omega_0$ and then multiplying it by half, as shown in Figure (2.22d).

### 2.3.6 Convolution:

The convolution of two functions $g(t)$ and $w(t)$, denoted by $g(t) * w(t)$, is defined by the integral

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(t)w(t - \tau)d\tau$$

The time convolution property and its dual, the frequency convolution property, state that if

$$g_1(t) \leftrightarrow G_1(\omega) \quad \text{and} \quad g_2(t) \leftrightarrow G_2(\omega)$$

Then (time convolution)

$$g_1(t) * g_2(t) \leftrightarrow G_1(\omega)G_2(\omega) \quad 2.43$$

And (frequency convolution)

$$g_1(t)g_2(t) \leftrightarrow G_1(\omega) * G_2(\omega) \quad 2.44$$

**Bandwidth of the Product of Two Signals:**

If $g_1(t)$ and $g_2(t)$ have bandwidths $B_1$ and $B_2$ Hz, respectively, the bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz. This result follows from the application of the width property of convolution to equation (2.44). This property states that the width of $x * y$ is the sum of the widths of $x$ and $y$. Consequently, if the bandwidth of $g(t)$ is $B$ Hz, then the bandwidth of $g^2(t)$ is $2B$ Hz, and the bandwidth of $g^3(t)$ is $nB$ Hz.
2.3.7 Time Differentiation and Time Integration

If

\[ g(t) \leftrightarrow G(\omega) \]

Then (Time differentiation)

\[ \frac{d^n g}{dt^n} \leftrightarrow (j\omega)^n G(\omega) \quad 2.48 \]

And (Time integration)

\[ \int_{-\infty}^{t} g(\tau) d\tau \leftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0) \delta(\omega) \quad 2.47 \]

Example 2.15: Using the time differentiation property, find the Fourier transform of the triangle pulse \( \Delta(t/\tau) \) shown in Figure (2.25a).

Solution: To find the Fourier transform of this pulse we differentiate it successively, as shown in Figure (2.25b and c). The second derivative consists of a sequence of impulses, as shown in Figure (2.25c). Recall that the derivative of a signal at a jump discontinuity is an impulse of strength equal to the amount of jump.

![Figure 2.25](image)

Figure (2.25): Finding the Fourier transform of a piecewise-linear signal using the time differentiation property.

The function \( dg/dt \) has a positive jump of \( 2/\tau \) at \( t = \pm \tau/2 \), and a negative jump of \( 4/\tau \) at \( t = 0 \). Therefore,
\[
\frac{d^2 g}{dt^2} = \frac{2}{\tau} \left[ \delta \left( t + \frac{\tau}{2} \right) - 2\delta(t) + \delta \left( t - \frac{\tau}{2} \right) \right] \quad 2.49
\]

From the time differentiation property (2.48),

\[
\frac{d^2 g}{dt^2} \leftrightarrow (j\omega)^2 G(\omega) = -\omega^2 G(\omega) \quad 2.50a
\]

Also, from the time-shifting property (2.30),

\[\delta(t - t_0) \leftrightarrow e^{-j\omega t_0} \quad 2.50b\]

Taking the Fourier transform of equation (2.49) and using the results in equations (2.50), we obtain

\[
-\omega^2 G(\omega) = \frac{2}{\tau} \left( e^{j\omega \frac{\tau}{2}} - 2 + e^{-j\omega \frac{\tau}{2}} \right) = \frac{4}{\tau} \left( \cos \frac{\omega \tau}{2} - 1 \right) = -\frac{8}{\tau} \sin^2 \left( \frac{\omega \tau}{4} \right)
\]

And

\[
G(\omega) = \frac{8}{\omega^2 \tau} \sin^2 \left( \frac{\omega \tau}{4} \right) = \frac{\tau}{2} \left[ \sin \left( \frac{\omega \tau}{4} \right) \right]^2 = \frac{\tau}{2} \text{sinc}^2 \left( \frac{\omega \tau}{4} \right) \quad 2.51
\]

The spectrum \(G(\omega)\) is shown in Figure (2.25d).

### 2.4 Signal Transmission Through a Linear System:

For a linear, time-invariant, continuous-time system the input-output relationship is given by

\[
y(t) * h(t) \quad 2.52
\]

Where \(g(t)\) is the input, \(y(t)\) is the output, and \(h(t)\) is the unit impulse response of the linear time-invariant system. If

\[
g(t) \leftrightarrow G(\omega), \quad y(t) \leftrightarrow Y(\omega), \quad \text{and} \quad h(t) \leftrightarrow H(\omega)
\]

Where \(H(\omega)\) is the system transfer function, then application of the time convolution property to equation (2.52) yields

\[
Y(\omega) = G(\omega)H(\omega) \quad 2.53
\]
2.4.1 Signal Distortion During Transmission

The transmission of an input signal $g(t)$ through a system changes it into the output signal $y(t)$. Equation (2.53) shows the nature of this change or modification. Here $G(\omega)$ and $Y(\omega)$ are the spectra of the input and the output, respectively. Therefore, $H(\omega)$ is the spectral response of the system. Equation (2.53) can be expressed in polar form as

$$|Y(\omega)|e^{j\theta_y(\omega)} = |G(\omega)||H(\omega)|e^{j[\theta_g(\omega)+\theta_h(\omega)]}$$

Therefore,

$$|Y(\omega)| = |G(\omega)||H(\omega)| \quad 2.54a$$

$$\theta_y(\omega) = \theta_g(\omega) + \theta_h(\omega) \quad 2.54b$$

During the transmission, the input signal amplitude spectrum $|G(\omega)|$ is changed to $|G(\omega)||H(\omega)|$. Similarly, the input signal phase spectrum $\theta_g(\omega)$ is changed to $\theta_g(\omega) + \theta_h(\omega)$. The frequency response of the system is $H(\omega)$. During transmission through the system, some frequency components may be boosted in amplitude, while others may be attenuated. The relative phases of the various components also changed. In general, the output waveform will be different from the input waveform.

**Distortionless Transmission:**

In several applications, such as signal amplification or message signal transmission over a communication channel, we require the output waveform to be a replica of the input waveform. In such cases, we need to minimize the distortion caused by the amplifier or the communication channel. It is therefore of practical interest to determine the characteristics of a system that allows a signal to pass without distortion (distortionless transmission).

Transmission is said to be distortionless if the input and the output have identical wave shapes within a multiplicative constant. A delayed output that retains the input waveform is also considered distortionless. Thus, in distortionless transmission, the input $g(t)$ and the output $y(t)$ satisfy the condition

$$y(t) = kg(t - t_d) \quad 2.55$$

The Fourier transform of this equation yields

$$Y(\omega) = kG(\omega)e^{-j\omega t_d}$$
But

\[ Y(\omega) = G(\omega)H(\omega) \]

Therefore,

\[ H(\omega) = ke^{-j\omega t_d} \]

This is the transfer function required for distortionless transmission. From this equation it follows that

\[ |H(\omega)| = k \quad 2.56a \]

\[ \theta_h(\omega) = -\omega t_d \quad 2.56b \]

This shows that for distortionless transmission, the amplitude response \(|H(\omega)|\) must be a constant, and the phase response \(\theta_h(\omega)\) must be a linear function of \(\omega\), as shown in Figure (2.26). The slope of \(\theta_h(\omega)\) with respect to \(\omega\) is \(-t_d\), where \(t_d\) is the delay of the output with respect to the input.

The time delay resulting from the signal transmission through a system is the negative of the slope of the system phase response \(\theta_h\); that is,

\[ t_d(\omega) = -\frac{d\theta_h}{d\omega} \quad 2.57 \]

\[ \text{Figure (2.26): Linear time-invariant system frequency response for distortionless transmission.} \]
Example 2.16: If \( g(t) \) and \( y(t) \) are the input and the output, respectively, of a simple RC low-pass filter as shown in Figure (2.27a), determine the transfer function \( H(\omega) \) and sketch \( |H(\omega)| \), \( \theta_h(\omega) \), and \( t_d(\omega) \). For distortionless transmission through this filter, what is the requirement on the bandwidth of \( g(t) \) if amplitude response variation within 2\% and time delay variation within 5\% are tolerable? What is the transmission delay? Find the output \( y(t) \).

Figure (2.27): Simple RC filter, its frequency response and time delay.

Solution: Application of the voltage division rule to this circuit yields

\[
H(\omega) = \frac{1/(j\omega C)}{R + 1/(j\omega C)} = \frac{1}{1 + j\omega RC} = \frac{a}{a + j\omega}
\]

Where
\[ a = \frac{1}{RC} = 10^6 \]

Hence,

\[ |H(\omega)| = \frac{a}{\sqrt{a^2 + \omega^2}} \approx 1 \quad \omega \ll a \]

\[ \theta_h(\omega) = -\tan^{-1} \frac{\omega}{a} \approx -\frac{\omega}{a} \quad \omega \ll a \]

Finally, the time delay is

\[ t_d(\omega) = -\frac{d \theta_h}{d \omega} = \frac{a}{\omega^2 + a^2} \approx \frac{1}{a} = 10^{-6} \quad \omega \ll a \]

The amplitude and phase response characteristics are given in Figure (2.27b). The time delay \( t_d \) as a function of \( \omega \) is shown in Figure (2.27c). For \( \omega \ll a \) (\( a = 10^6 \)), the amplitude response is practically constant and the phase shift is nearly linear. The phase linearity results in a constant time delay characteristics. The filter therefore can transmit low-frequency signals with negligible distortion.

In our case, amplitude response variation within 2\% and time delay variation within 5\% are tolerable. Let \( \omega_0 \) be the highest bandwidth of a signal that can be transmitted within these specifications. To compute \( \omega_0 \) observe that the filter is a low-pass filter with gain and time delay both at maximum when \( \omega = 0 \) and

\[ |H(0)| = 1 \quad and \quad t_d(0) = \frac{1}{a} \]

Therefore, \( |H(\omega_0)| \geq 0.98 \) and \( t_d(\omega_0) \geq 0.95/a \), so that

\[ |H(\omega_0)| = \frac{a}{\sqrt{\omega_0^2 + a^2}} \geq 0.98 \rightarrow \omega_0 \leq 0.203a = 203,000 \]

\[ t_d(\omega_0) = \frac{a}{\omega_0^2 + a^2} \geq \frac{0.95}{a} \rightarrow \omega_0 \leq 0.2294a = 229,400 \]

The smaller of the two values, \( \omega_0 = 203000 \) rad/s or 32.31 kHz, is the highest bandwidth that satisfies both constraints on \( |H(\omega)| \) and \( t_d \). The time delay \( t_d \approx \frac{1}{a} = 10^{-6} \) over this band [see figure (2.27c)]. Also the amplitude response is almost unity [figure (2.27b)]. Therefore, the output \( y(t) \approx g(t - 10^{-6}) \).
Ideal and Practical Filters:

Ideal filters allow distortionless transmission of a certain band of frequencies and suppress all the remaining frequencies. The ideal low-pass filter as shown in Figure (2.28), for example, allows all components below $\omega = W$ rad/s to pass without distortion and suppresses all components above $\omega = W$.

![Ideal low-pass filter frequency response and its impulse response.](image)

Figure (2.28): Ideal low-pass filter frequency response and its impulse response.

Figure (2.29) shows ideal high-pass and bandpass filter characteristics.

![Ideal high-pass and band-pass filter frequency responsees.](image)

Figure (2.29): Ideal high-pass and band-pass filter frequency responsees.

The ideal low-pass filter in Figure (2.28a) has a linear phase of slope $-t_d$, which results in a time delay of $t_d$ seconds for all its input components of frequencies below $W$ rad/s. Therefore, if the input is a signal $g(t)$ band-limited to $W$ rad/s, the output $y(t)$ is $g(t)$ delayed by $t_d$, that is,

$$y(t) = g(t - t_d)$$
The signal $g(t)$ is transmitted by this system without distortion, but with time delay $t_d$. For this filter $|H(\omega)| = \text{rect}(\omega/2W)$, and $\theta_h(\omega) = -\omega t_d$, so that

$$|H(\omega)| = \text{rect}\left(\frac{\omega}{2W}\right)e^{-j\omega t_d} \quad 2.58a$$

The unit impulse response $h(t)$ of this filter is found from pair 18 in table (2.1) and the time-shifting property:

$$h(t) = F^{-1}\left[\text{rect}\left(\frac{\omega}{2W}\right)e^{-j\omega t_d}\right]$$

$$= \frac{W}{\pi}\text{sinc}[W(t - t_d)] \quad 2.58b$$

physically this system is not realizable. For a physically realizable system, $h(t)$ must be causal; that is,

$$h(t) = 0 \quad \text{for } t < 0$$

the impulse response $h(t)$ in figure (2.28) is not realizable. One practical approach to filter design is to cut off the tail of $h(t)$ for $t < 0$. The resulting casual impulse response $\hat{h}(t)$, where

$$\hat{h}(t) = h(t)u(t)$$

is physically realizable because it is causal. If $t_d$ is sufficiently large, $\hat{h}(t)$ will be a close approximation of $h(t)$, and the resulting filter $\hat{H}(\omega)$ will be a good approximation of an ideal filter.

In practice, we can realize a variety of filter characteristics to approach ideal characteristics. Practical filter characteristics are gradual, without jump discontinuities in the amplitude response $|H(\omega)|$. The well-known Butterworth filters, for example, have amplitude response

$$|H(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{2\pi B}\right)^{2n}}}$$

These characteristics are shown in Figure (2.31) for several values of $n$ (the order of the filter). Note that the amplitude response approaches an ideal low-pass behavior as $n \to \infty$. 

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Digital Filters:

Analog signals can also be processed by digital means (A/D conversion). This involves sampling, quantizing, and coding. The resulting digital signal can be processed by a small, special-purpose digital computer designed to convert the input sequence into a desired output sequence. The output sequence is converted back into the desired analog signal. A special algorithm of the processing digital computer can be used to achieve a given signal operation (e.g., low-pass, bandpass, or high-pass filtering).

Digital processing of analog signals has several advantages. A small, special-purpose computer can be time-shared for several uses, and the cost of digital implementation is often considerably lower than that of its analog counterpart. The accuracy of a digital filter is dependent only on the computer word length, the quantizing interval, and the sampling rate. Digital filters employ simple elements, such as adders, multipliers, shifters, and delay elements, rather than RLC components and operational amplifiers. As a result, they are generally unaffected by such factors as component accuracy, temperature stability, long-term drift, and so on, that afflict analog filter circuits. Also many of the circuit restrictions imposed by physical limitations of analog devices can be removed, or at least circumvented, in a digital processor. Moreover, filters of a high order can be realized easily. Finally, digital filters can be modified simply by changing the algorithm of the computer, in contrast to an analog system, which may have to be physically rebuilt.
Signal Distortion Over a Communication Channel.

A signal transmitted over a channel is distorted because of various channel imperfections. The nature of signal distortion will now be studied.

Linear Distortion:

We shall first consider linear—time invariant channels. Signal distortion can be caused over such a channel by nonideal characteristics of either the magnitude, the phase, or both. We can identify the effects these nonidealities will have on a pulse \( g(t) \) transmitted through such a channel.

Let the pulse exist over the interval \((a, b)\) and be zero outside this interval. The components of the Fourier spectrum of the pulse have such a perfect and delicate balance of magnitudes and phases that they add up precisely to the pulse \( g(t) \) over the interval \((a, b)\) and to zero outside this interval. The transmission of \( g(t) \) through an ideal channel that satisfies the conditions of a distortionless transmission also leaves this balance undisturbed, because a distortionless channel multiplies each component by the same factor and delays each component by the same amount of time. Now, if the amplitude response of the channel is not ideal [that is, \( |H(\omega)| \) is not equal to a constant], this delicate balance will be disturbed, and the sum of all the components cannot be zero outside the interval \((a, b)\). In short, the pulse will spread out. The same thing happens if the channel phase characteristic is not ideal, that is, \( \theta_h(\omega) \neq -\omega t_d \). Thus, spreading, or dispersion, of the pulse will occur if either the amplitude response or the phase response, or both, are nonideal.

This type of distortion is undesirable in a TDM system, because pulse spreading causes interference with a neighboring pulse and consequently with a neighboring channel (crosstalk). For an FDM system, this type of distortion causes distortion (dispersion) in each multiplexed signal, but no interference occurs with a neighboring channel. This is because in FDM, each of the multiplexed signals occupies a band not occupied by any other signal. The amplitude and phase nonidealities of a channel will distort the spectrum of each signal, but because they are all nonoverlapping, no interference occurs among them.

Example 2.17: A low-pass filter (Figure (2.33a)) transfer function \( H(\omega) \) is given by

\[
H(\omega) = \begin{cases} 
(1 + k \cos T \omega) e^{-j\omega t_d} & |\omega| < 2\pi B \\
0 & |\omega| > 2\pi B 
\end{cases}
\]

2.60

A pulse \( g(t) \) band-limited to \( B \) Hz (figure (2.33b)) is applied at the input of this filter. Find the output \( y(t) \).
Solution: This filter has ideal phase and nonideal magnitude characteristics. Because\[ g(t) \leftrightarrow G(\omega), \quad y(t) \leftrightarrow Y(\omega) \]and\[ Y(\omega) = G(\omega)H(\omega) \]
\[ = G(\omega)(1 + k \cos T\omega)e^{-j\omega t_d} \]
\[ = G(\omega)e^{-j\omega t_d} + k[G(\omega) \cos T\omega]e^{-j\omega t_d} \quad 2.61 \]

Using the time-shifting property and Equations (2.30a) and (2.32), we have\[ y(t) = g(t - t_d) + \frac{k}{2}[g(t - t_d - T) + g(t - t_d + T)] \quad 2.62 \]
The output consists of \( g(t) \) and its echoes shifted by \( \pm t_d \).

Distortion Caused by Channel Nonlinearities

Until now we considered the channel to be linear. This approximation is valid only for small signals. For large amplitudes, nonlinearities cannot be ignored. Now we shall consider a simple case of a memoryless nonlinear channel where the input \( g \) and the output \( y \) are related by some nonlinear equation,

\[ y = f(g) \]

The right-hand side of this equation can be expanded in a McLaurin’s series as

\[ y(t) = a_0 + a_1 g(t) + a_2 g^2(t) + a_3 g^3(t) + \cdots + a_k g^k(t) + \cdots \]
We remember the result of convolution that if the bandwidth of $g(t)$ is $B$ Hz, then the bandwidth of $g^k(t)$ is $kB$ Hz. Hence, the bandwidth of $y(t)$ is $kB$ Hz. Consequently, the output spectrum spreads well beyond the input spectrum, and the output signal contains new frequency components not contained in the input signal. In broadcast communication, we need to amplify signals at very high power levels, where high-efficiency amplifiers (class C) are desirable. Unfortunately, these amplifiers are nonlinear, and their use to amplify signals causes distortion. This is one of the serious problems in AM signals. However, FM signals are not affected by nonlinear distortion. If a signal is transmitted over a nonlinear channel, the nonlinearity not only distorts the signal, but also causes interference with other signals on the channel because of its spectral dispersion (spreading). The spectral dispersion will cause a serious interference problem in FDM systems (but not in TDM systems).

**Example 2.18:** The input $x(t)$ and the output $y(t)$ of a certain nonlinear channel are related as

$$y(t) = x(t) + 0.001x^2(t)$$

Find the output signal $y(t)$ and its spectrum $Y(\omega)$ if the input signal is $x(t) = (1000/\pi) \text{sinc}(1000t)$. Verify that the bandwidth of the output signal is twice that of the input signal. This is the result of signal squaring. Can the signal $x(t)$ be recovered (without distortion) from the output $y(t)$?

**Solution:** Since

$$x(t) = \frac{1000}{\pi} \text{sinc}(1000t)$$

$$X(\omega) = \text{rect}\left(\frac{\omega}{2000}\right)$$

Where we have

$$y(t) = x(t) + 0.001x^2(t) = \frac{1000}{\pi} \text{sinc}(1000t) + \frac{1000}{\pi^2} \text{sinc}^2(1000t)$$

$$Y(\omega) = \text{rect}\left(\frac{\omega}{2000}\right) + 0.316\Delta\left(\frac{\omega}{4000}\right)$$

Note that $0.316\text{sinc}^2(1000t)$ is the unwanted (distortion) term in the received signal. Figure (2.34a) shows the input (desired) signal spectrum $X(\omega)$; Figure (2.34b) shows the spectrum of the undesired (distortion) term; and Figure (2.34c) shows the received signal spectrum $Y(\omega)$. See the following observations:
Figure (2.34): Signal distortion caused by nonlinear operation. (a) Desired (input) signal spectrum. (b) Spectrum of the unwanted signal (distortion) in the received signal. (c) Spectrum of the received signal. (d) Spectrum of the received signal after low-pass filtering.

1. The bandwidth of the received signal $y(t)$ is twice that of the input signal $x(t)$ (because of the signal squaring).

2. The received signal contains the input signal $x(t)$ plus an unwanted signal $1000/\pi \text{sinc}^2(1000t)$. The spectra of these two signals are shown in Figure (2.34a and b). Figure (2.34c) shows $Y(\omega)$, the spectrum of the received signal. Note that the desired signal and the distortion signal spectra overlap, and it is impossible to recover the signal $x(t)$ from the received signal $y(t)$ without some distortion.

3. We can reduce the distortion by passing the received signal through a low-pass filter of bandwidth 1000 rad/s. The spectrum of the output of this filter is shown in Figure (2.34d).

4. We have an additional problem of interference with other signals if the input signal $x(t)$ is frequency-division multiplexed (FDM) along with several other signals on this channel. This means that several signals occupying nonoverlapping frequency bands are transmitted simultaneously on the same channel. Spreading of the spectrum $X(\omega)$ outside its original band of 1000 rad/s will interfere with the
signal in the band of 1000 to 2000 rad/s. Thus, in addition to the distortion of \( x(t) \), we also have an interference with the neighboring band.

5. If \( x(t) \) were a digital signal consisting of a pulse train, each pulse would be distorted, but there would be no interference with the neighboring pulses. Moreover, even with distorted pulses, data can be received without loss because digital communication can withstand considerable pulse distortion without loss of information. Thus, if this channel were used to transmit a TDM signal consisting of two interleaved pulse trains, the data in the two trains would be recovered at the receiver.

**Distortion Caused by Multipath Effects:**

A multipath transmission takes place when a transmitted signal arrives at the receiver by two or more paths of different delays. For example, if a signal is transmitted over a cable that has impedance irregularities (mismatching) along the path, the signal will arrive at the receiver in the form of a direct wave plus various reflections with various delays. In radio links, the signal can be received by direct path between the transmitting and the receiving antennas and also by reflections from other objects, such as hills, buildings, and so on. In long-distance radio links using the ionosphere, similar effects occur because of one-hop and multihop paths. In each of these cases, the transmission channel can be represented as several channels in parallel, each with a different relative attenuation and a different time delay. Let us consider the case of only two paths: one with a unity gain and a delay \( t_d \), and the other with a gain \( \alpha \) and a delay \( t_d + \Delta t \), as shown in Figure (2.35a).

![Figure (2.35): Multipath transmission.](image-url)
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The transfer function of the two paths are given by \( e^{-j\omega t_d} \) and \( \alpha e^{-j\omega (t_d + \Delta t)} \), respectively. The overall transfer function of such a channel is \( H(\omega) \), given by

\[
H(\omega) = e^{-j\omega t_d} + \alpha e^{-j\omega (t_d + \Delta t)}
\]

\[
= e^{-j\omega t_d} \left( 1 + \alpha e^{-j\omega \Delta t} \right)
\]

\[
= e^{-j\omega t_d} \left( 1 + \alpha \cos \omega \Delta t - j\alpha \sin \omega \Delta t \right)
\]

\[
= \sqrt{1 + \alpha^2 + 2\alpha \cos \omega \Delta t} e^{-j\left(\omega t_d + \tan^{-1}\left(\frac{\alpha \sin \omega \Delta t}{1 + \alpha \cos \omega \Delta t}\right)\right)}
\]

Both the magnitude and the phase characteristics of \( H(\omega) \) are periodic in \( \omega \) with a period of \( 2\pi/\Delta t \) as shown in Figure (2.35b). The multipath transmission, therefore, causes nonidealities in the magnitude and the phase characteristics of the channel and will cause linear distortion (pulse dispersion).

If for instance, the gain of the two paths are very close, that is, \( \alpha \approx 1 \), the signals received by the two paths can very nearly cancel each other at certain frequencies, where their phases are \( \pi \) rad apart. Equation (2.63b) shows that at frequencies where \( \omega = n\pi/\Delta t \) (\( n \) odd), \( \cos \omega \Delta t = 1 \), and \( |H(\omega)| \approx 0 \) when \( \alpha \approx 1 \). These frequencies are the multipath null frequencies. At frequencies \( \omega = n\pi/\Delta t \) (\( n \) even), the two signals interfere constructively to enhance the gain. Such channels cause frequency-selective fading of transmitted signals. Such distortion can be partly corrected by using the tapped delay-line equalizer.

**Fading Channels**

Thus far, the channel characteristics were assumed to be constant with time. In practice, we encounter channels whose transmission characteristics vary with time. These include troposcatter channels and channels using the ionosphere for radio reflection to achieve long—distance communication. The time variations of the channel properties arise because of semiperiodic and random changes in the propagation characteristics of the medium. The reflection properties of the ionosphere, for example, are related to meteorological conditions that change seasonally, daily, and even from hour to hour, much the same way as dose the weather. Periods of sudden storms also occur. Hence, the effective channel transfer function varies semiperiodically and randomly, causing random attenuation of the signal. This phenomenon is known as fading. One way to reduce the effects of fading is to use **automatic gain control** (AGC).
Fading may be strongly frequency dependent where different frequency components are affected unequally. Such fading is known as frequency—selective fading and can cause serious problems in communication. Multipath propagation can cause frequency—selective fading.

### 2.7 Signal Energy and Energy Spectral Density

The energy $E_g$ of a signal $g(t)$ is defined as the area under $|g(t)|^2$. We can also determine the signal energy from its Fourier transform $G(\omega)$ through Parseval’s theorem.

**Parseval’s Theorem:** Signal energy can be related to the signal spectrum $G(\omega)$ by substituting Equation (2.8b) in equation (1.2):

$$E_g = \int_{-\infty}^{\infty} g(t) g^*(t) dt = \int_{-\infty}^{\infty} g(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) e^{-j\omega t} d\omega \right] dt$$

Here, we used the fact that $g^*(t)$, being the conjugate of $g(t)$, can be expressed as the conjugate of the right-hand side of equation (2.8b). Now, interchanging the order of integration yields

$$E_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \left[ \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) G^*(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

This is the statement of the well—known Parseval’s theorem. A similar result was obtained for a periodic signal and its Fourier series. This result allows us to determine the signal energy from either the time—domain specification $g(t)$ or the frequency—domain specification $G(\omega)$ of the same signal.

**Example 2.19:** Verify Parseval’s theorem for the signal $g(t) = e^{-at} u(t)$ ($a > 0$).

**Solution:** we have

$$E_g = \int_{-\infty}^{\infty} g^2(t) dt = \int_{0}^{\infty} e^{-2at} dt = \frac{1}{2a}$$

We now determine $E_g$ from the signal spectrum $G(\omega)$ given by
And from equation (2.64),

\[ E_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{2\pi} \tan^{-1} \frac{\omega}{a} \bigg|_{-\infty}^{\infty} = \frac{1}{2a} \]

This is verifies Parseval’s theorem.

**Energy Spectral Density (ESD)**

Consider a signal \( g(t) \) applied at the input of an ideal bandpass filter, whose transfer function \( H(\omega) \) is shown in Figure (2.36a), this filter suppresses all frequencies except a narrow band \( \Delta \omega \) (\( \Delta \omega \to 0 \)) centered at frequency \( \omega_0 \) (Figure 2.36b).

If the filter output is \( y(t) \), then its Fourier transform \( Y(\omega) = G(\omega)H(\omega) \), and \( E_y \), the energy of the output \( y(t) \), is

\[ E_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)H(\omega)|^2 d\omega \quad 2.66 \]

Because \( H(\omega) = 1 \) over the passband \( \Delta \omega \), and zero everywhere else, the integral on the right—hand side is the sum of the two shaded areas in Figure (2.36b), and we have (for \( \Delta \omega \to 0 \))

\[ E_y = 2 \frac{1}{2\pi} |G(\omega_0)|^2 d\omega = 2|G(\omega_0)|^2 df \]
Thus, $2|G(\omega_0)|^2df$ is the energy contributed by the spectral components within the two narrow bands, each of width $\Delta f$ Hz, centered at $\pm \omega_0$. Hence, $|G(\omega_0)|^2$ is the energy spectral density (per unit bandwidth in hertz) of $g(t)$. Actually the energy contributed per unit bandwidth is $2|G(\omega_0)|^2$ because both the positive and the negative frequency components combine to form the components in the band $\Delta f$. However, for the sake of convenience we consider the positive and negative frequency components being independent. Some authors do define $2|G(\omega_0)|^2$ as the energy spectral density. The energy spectral density (ESD) $\psi_g(t)$ is thus defined as

$$\psi_g(\omega) = |G(\omega_0)|^2$$

And equation (2.64) can be expressed as

$$E_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_g(\omega)d\omega = \int_{-\infty}^{\infty} \psi_g(\omega)df$$

From the results in example 2.19, the ESD of the signal $g(t) = e^{-\alpha t}u(t)$ is

$$\psi_g(\omega) = |G(\omega_0)|^2 = \frac{1}{\omega^2 + \alpha^2}$$

**Essential Bandwidth of a Signal**

The spectra of most signals extend to infinity. However, because the energy of a practical signal is finite, the signal spectrum must approach 0 as $\omega \to \infty$. Most of the signal energy is contained within a certain band of $B$ Hz, and the energy content of the components of frequencies greater than $B$ Hz is negligible. To see that, let us see an example.

**Example 2.20:** Estimate the essential bandwidth $W$ rad/s of the signal $e^{-\alpha t}u(t)$ if the essential band is required to contain 95% of the signal energy.

**Solution:** In this case,

$$G(\omega) = \frac{1}{j\omega + \alpha}$$

And the ESD is

$$|G(\omega)|^2 = \frac{1}{\omega^2 + \alpha^2}$$

This ESD is shown in Figure (2.37).
Moreover, the signal energy $E_g$ is $1/2\pi$ times the area under this ESD, which has already been found to be $1/2a$. Let $W \text{ rad/s}$ be the essential bandwidth, which contains 95% of the total signal energy $E_g$. This means $1/2\pi$ times the shaded area in Figure (2.37) is $0.95/2a$, that is,

$$\frac{0.95}{2a} = \frac{1}{2\pi} \int_{-W}^{W} \frac{d\omega}{\omega^2 + a^2}$$

$$= \frac{1}{2\pi a} \left[ \tan^{-1} \frac{\omega}{a} \right]_{-W}^{W} = \frac{1}{\pi a} \tan^{-1} \frac{W}{a}$$

Or

$$\frac{0.95}{2} = \tan^{-1} \frac{W}{a} \rightarrow W = 12.706a \text{ rad/s}$$

This means that the spectral components of $g(t)$ in the band from 0 (dc) to 12.706 rad/s (2.02 Hz) contribute 95% of the total signal energy; all the remaining spectral components (in the band from 12.706 rad/s to $\infty$) contribute only 5% of the signal energy.

**Example 2.21:** Estimate the essential bandwidth of a rectangular pulse $g(t) = \text{rect} \left( \frac{t}{T} \right)$ (figure (2.38a), where the essential bandwidth is to contain at least 90% of the pulse energy.

**Solution:** For this pulse, the energy $E_g$ is

$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = T$$
Also because

\[
\text{rect} \left( \frac{t}{T} \right) \leftrightarrow T \text{sinc} \left( \frac{\omega T}{2} \right)
\]

The ESD for this pulse is

\[
\Psi_g(\omega) = |G(\omega)|^2 = T^2 \text{sinc}^2 \left( \frac{\omega T}{2} \right)
\]

This ESD is shown in figure (2.38b) as a function of \(\omega T\) as well as \(f T\), where \(f\) is the frequency in hertz. The energy \(E_W\) within the band from 0 to \(W\) rad/s is given by

\[
E_W = \frac{1}{2\pi} \int_{-W}^{W} T^2 \text{sinc}^2 \left( \frac{\omega T}{2} \right) d\omega
\]

Setting \(\omega T = x\) in this integral so that \(d\omega = (1/T)dx\), we obtain

\[
E_W = \frac{T}{\pi} \int_{0}^{WT} \text{sinc}^2 \left( \frac{x}{2} \right) dx
\]
Also because $E_g = T$, we have

$$\frac{E_W}{E_g} = \frac{1}{\pi} \int_0^{WT} \text{sinc}^2 \left( \frac{x}{2} \right) dx$$

The integral on the right—hand side is numerically computed, and the plot of $\frac{E_W}{E_g}$ vs. $WT$ is shown in Figure (2.38c). Note that 90.28% of the total energy of the pulse $g(t)$ is contained within the band $W = 2\pi/T$ $\text{rad}$ $s$ or $B = 1/T$ $\text{Hz}$. Therefore, using the 90% criterion, the bandwidth of a rectangular pulse of width $T$ seconds is $1/T$ $\text{Hz}$.

**Energy of Modulated Signals**

We have seen that modulation shifts the signal spectrum $G(\omega)$ to the left and right by $\omega_0$. We now show that a similar thing happens to the ESD of the modulated signal.

Let $g(t)$ be a baseband signal band-limited to $B$ $\text{Hz}$. The amplitude—modulated signal $\varphi(t)$ is

$$\varphi(t) = g(t) \cos \omega_0 t$$

And the spectrum (Fourier transform) of $\varphi(t)$ is

$$\Phi(\omega) = \frac{1}{2} [G(\omega + \omega_0) + G(\omega - \omega_0)]$$

The ESD of the modulated signal $\varphi(t)$ is $|\Phi(\omega)|^2$, that is,

$$\psi_\varphi(\omega) = \frac{1}{4} |G(\omega + \omega_0) + G(\omega - \omega_0)|^2$$

If $\omega_0 \geq 2\pi B$, then $G(\omega + \omega_0)$ and $G(\omega - \omega_0)$ are nonoverlapping, and

$$\psi_\varphi(\omega) = \frac{1}{4} [G(\omega + \omega_0)]^2 + [G(\omega - \omega_0)]^2 \tag{2.69a}$$

$$= \frac{1}{4} [\psi_g(\omega + \omega_0) + \psi_g(\omega - \omega_0)] \tag{2.69b}$$

Because of the energy of a signal is proportional to the area under the ESD, it follows that the energy of $\varphi(t)$ is half the energy of $g(t)$, that is,

$$E_\varphi = \frac{1}{2} E_g \quad \omega_0 \geq 2\pi B \tag{2.70}$$
**Time Autocorrelation Function and the Energy Spectral Density**

In lecture 1, we showed that a good measure of comparing two signals \( g(t) \) and \( z(t) \) is the correlation function \( \psi_{gz}(\tau) \). We also defined the correlation of a signal \( g(t) \) with itself (autocorrelation). For a real signal \( g(t) \), the autocorrelation function \( \psi_g(\tau) \) is given by

\[
\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t + \tau)dt \tag{2.71a}
\]

Setting \( x = t + \tau \) in equation (2.71a) yields

\[
\psi_g(\tau) = \int_{-\infty}^{\infty} g(x)g(x - \tau)dx
\]

In this equation, \( x \) is a dummy variable and could be replaced by \( t \). Thus,

\[
\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t + \tau)dt \tag{2.71b}
\]

This shows that for a real \( g(t) \), the autocorrelation function is an even function of \( \tau \), that is,

\[
\psi_g(\tau) = \psi_g(-\tau) \tag{3.72}
\]

We see now that the ESD \( \psi_g(\omega) = |G(\omega)|^2 \) is the Fourier transform of the autocorrelation function \( \psi_g(\tau) \). That is,

\[
\psi_g(\tau) \leftrightarrow \psi_g(\omega) = |G(\omega)|^2 \tag{2.73}
\]

A careful observation of the operation of correlation show close connection to convolution. Indeed, the autocorrelation function \( \psi_g(\tau) \) is the convolution of \( g(\tau) \) with \( g(-\tau) \).

**Example 2.22:** Find the time autocorrelation function of the signal \( g(t) = e^{-at}u(t) \), and from it determine the ESD of \( g(t) \).

**Solution:** In this case,

\[
g(t) = e^{-at}u(t) \quad \text{and} \quad g(t - \tau) = e^{-a(t-\tau)}u(t - \tau)
\]

Recall that \( g(t - \tau) \) is \( g(t) \) right—shifted by \( \tau \), as shown in Figure (2.40a) (for positive \( \tau \)). The autocorrelation function \( \psi_g(\tau) \) is given by the area under the product \( g(t)g(t - \tau) \). Therefore,
\[
\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt = e^{a\tau} \int_{\tau}^{\infty} e^{-2at} dt = \frac{1}{2a} e^{-a\tau}
\]

This is valid for positive \(\tau\). We can perform a similar procedure for negative \(\tau\). However, we know that for a real \(g(t)\), \(\psi_g(\tau)\) is an even function of \(\tau\). Therefore,

\[
\psi_g(\tau) = \frac{1}{2a} e^{-a|\tau|}
\]

Figure (2.40b) shows the autocorrelation function \(\psi_g(\tau)\). The ESD \(\psi_g(\omega)\) is the Fourier transform of \(\psi_g(\tau)\). From table 3.1 (pair 3), it follows that

\[
\psi_g(\omega) = \frac{1}{\omega^2 + a^2}
\]

**ESD of the Input and the Output:** If \(g(t)\) and \(y(t)\) are the input and the correspondingly output of a linear time—invariant (LTI) system, then

\[
Y(\omega) = H(\omega)G(\omega)
\]

Therefore,

\[
|Y(\omega)|^2 = |H(\omega)|^2|G(\omega)|^2
\]

This shows that

\[
\psi_y(\omega) = |H(\omega)|^2\psi_g(\omega)
\]

Thus, the output signal ESD is \(|H(\omega)|^2\) times the input signal ESD.
2.8 Signal Power and Power Spectral Density

For a power signal, a meaningful measure of its size is its power as the time average of the signal energy averaged over the infinite time interval. The power $P_g$ of a real signal $g(t)$ is given by

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t) dt$$  \hspace{1cm} (2.75)

The signal power and the related concepts can be readily understood by defining a truncated signal $g_T(t)$ as

$$g_T(t) = \begin{cases} g(t) & |t| \leq T/2 \\ 0 & |t| > T/2 \end{cases}$$

The truncated signal is shown in Figure (2.41). The integral on the right-hand side of equation (2.75) is the energy of the truncated signal $g_T(t)$, thus,

$$P_g = \lim_{T\to\infty} \frac{E_{g_T}}{T}$$  \hspace{1cm} (2.76)

This equation serves as a link between power and energy.

![Figure (2.41): Limiting process in derivation of PSD.](image)

**Power Spectral Density (PSD)**

If the signal $g(t)$ is a power signal, then its power is finite, and the truncated signal $g_T(t)$ is an energy signal as long as $T$ is finite. If $g_T(t)$ is an energy signal as long as $T$ is finite, and $g_T(t) \leftrightarrow G_T(\omega)$, we define the **Power Spectral Density (PSD)** $S_g(\omega)$ as

$$S_g(\omega) = \lim_{T\to\infty} \frac{|G_T(\omega)|^2}{T}$$  \hspace{1cm} (2.79)

Consequently,

$$P_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_g(\omega) d\omega$$  \hspace{1cm} (2.80a)
This equation can be expressed in a more compact form using the variable \( f \) (in Hz) as

\[
P_g = \int_{-\infty}^{\infty} S_g(\omega) \, d\omega = 2\int_{0}^{\infty} S_g(\omega) \, df
\]

**Time—Autocorrelation Function of Power Signals**

The (time) autocorrelation function \( R_g(\tau) \) of a real power signal \( g(t) \) is defined as

\[
R_g(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t+\tau) \, dt
\]

Usign the same argument as that used for energy signals [equations (2.71b) and (2.72)], we can show that \( R_g(\tau) \) is an even function of \( \tau \). This means for a real \( g(t) \)

\[
R_g(\tau) = R_g(-\tau)
\]

And

For energy signals, the ESD \( \Psi_g(\omega) \) is the Fourier transform of the autocorrelation function \( \psi_g(\tau) \). A similar result applies to power signals. We now show that for a power signal, the PSD \( S_g(\omega) \) is the Fourier transform of the autocorrelation function \( R_g(\tau) \).

From equation (2.82a) and Figure (2.41),

\[
R_g(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} g_T(t)g_T(t+\tau) \, dt = \lim_{T \to \infty} \frac{\psi_{g_T}(\tau)}{T}
\]

Recall that \( \psi_{g_T}(\tau) \leftrightarrow |G_T(\omega)|^2 \). Hence, the Fourier transform of the preceding equation yields

\[
R_g(\tau) \leftrightarrow \lim_{T \to \infty} \frac{|G_T(\omega)|^2}{T} = S_g(\omega)
\]

The concept and relationships for signal power are parallel to those for signal energy. This is brought out in Table (2.3).

**Table 2.3:**
Example 2.23: Figure (2.42a) shows a random binary pulse train $g(t)$. The pulse width is $T_b/2$, and one binary digit is transmitted every $T_b$ seconds. A binary 1 is transmitted by the positive pulse, and a binary 0 is transmitted by the negative pulse. The two symbols are equally likely and occur randomly. Determine the autocorrelation function, the PSD, and the essential bandwidth of this signal.

Solution: We cannot describe this signal as a function of time because the precise waveform is not known due to its random nature. We do, however, know its behavior in terms of the averages (statistical information). The autocorrelation function, being an average parameter (time average) of the signal, is determined from the given statistical (average) information. We have

$$R_g(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t-\tau)dt$$

Figure (2.42b) shows $g(t)$ by solid lines and $g(t-\tau)$, which is $g(t)$ delayed by $\tau$, by dashed lines. To determine the integrand on the right—hand side of the above equation, we multiply $g(t)$ with $g(t-\tau)$, find the area under the product $g(t)g(t-\tau)$, and divide it by the averaging interval $T$. Let there be $N$ bits (pulses) during this interval $T$ so that $T = NT_b$, and as $T \to \infty$, $N \to \infty$. Thus,

$$R_g(\tau) = \lim_{N \to \infty} \frac{1}{NT_b} \int_{-NT_b/2}^{NT_b/2} g(t)g(t-\tau)dt$$

Let us consider first the case of $\tau < T_b/2$. In this case there is an overlap (shown by the dashed region) between each pulse of $g(t)$ and that of $g(t-\tau)$. 
The area under the product $g(t)g(t - \tau)$ is $T_b/2 - \tau$ for each pulse. Since there are $N$ pulses during the averaging interval, then the total area under $g(t)g(t - \tau)$ is $N(T_b/2 - \tau)$, and

$$R_g(\tau) = \lim_{N \to \infty} \frac{1}{N T_b} \left[ N \left( \frac{T_b}{2} - \tau \right) \right]$$

$$= \frac{1}{2} \left( 1 - \frac{2\tau}{T_b} \right) \quad \tau < \frac{T_b}{2}$$

Because $R_g(\tau)$ is an even function of $\tau$, 

\[ \text{Figure (2.42): Autocorrelation function and power spectral density function of a random binary pulse train.} \]
As shown in figure (2.42c).

As we increase $\tau$ beyond $T_b/2$, there will be overlap between each pulse and its immediate neighbor. The two overlapping pulses are equally likely to be of the same polarity or of opposite polarity. Their product is equally likely to be 1 or $-1$ over the overlapping interval. On the average, half the pulse products will be 1 (positive—positive or negative—positive combinations). Consequently, the area under $g(t)g(t-\tau)$ will be zero when averaged over an infinitely large time ($T \to \infty$), and

$$R_g(\tau) = \begin{cases} 0 & |\tau| > \frac{T_b}{2} \\ \frac{1}{2} \left( 1 - \frac{2|\tau|}{T_b} \right) & |\tau| < \frac{T_b}{2} \end{cases} \quad 2.88a$$

The autocorrelation function in this case is the triangle function $\frac{1}{2} \Delta(t/T_b)$ shown in figure (2.42c). The PSD is the Fourier transform of $\frac{1}{2} \Delta(t/T_b)$, which is found in example (2.15) or table (2.1 pair 19) as

$$S_g(\omega) = \frac{T_b}{4} \text{sinc}^2 \left( \frac{\omega T_b}{4} \right) \quad 2.89$$

The PSD is the square of the sinc function shown in Figure (2.42d). From the result in example (2.21), we conclude that the 90.28% of the area of this spectrum is contained within the band from $0$ to $4\pi/T_b$ rad/s, or from $0$ to $2/T_b$ Hz. Thus, the essential bandwidth may be taken as $2/T_b$ Hz (assuming a 90% power criterion). This example illustrates dramatically how the autocorrelation function can be used to obtain the spectral information of a (random) signal where convolution means of obtaining the Fourier spectrum are not usable.

**Input and Output Power Spectral Densities**

We can readily show that if $g(t)$ and $y(t)$ are the input and output signals of an LTI system with transfer function $H(\omega)$, then

$$S_y(\omega) = |H(\omega)|^2 S_g(\omega) \quad 2.90$$

**Example 2.24:** A noise signal $n_1(t)$ with PSD $S_{n_1}(\omega) = K$ is applied at the input of an ideal differentiator [Figure (2.43a)]. Determine the PSD and the power of the output noise signal $n_0(t)$.

**Solution:** The transfer function of an ideal differentiator is $H(\omega) = j\omega$. If the noise at the demodulator output is $n_0(t)$, then from equation (2.90),
\[ S_{n_0}(\omega) = |H(\omega)|^2 S_n(\omega) = |j\omega|^2 K \]

The output PSD \( S_{n_0}(\omega) \) is parabolic, as shown in Figure (2.43c). The output noise power \( N_0 \) is \( 1/2\pi \) times the area under the output PSD. Therefore,

\[ N_0 = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} K \omega^2 d\omega = K \int_{-2\pi B}^{2\pi B} \omega^2 d\omega = \frac{8\pi^2 B^3 K}{3} \]

**Figure (2.43): Power spectral density at the input and the output of an ideal differentiator.**

**PSD of a Modulated Signals**

For a power signal \( g(t) \), if

\[ \varphi(t) = g(t) \cos \omega_0 t \]

Then the PSD \( S_\varphi(\omega) \) of the modulated signal \( \varphi(t) \) is given by

\[ S_\varphi(\omega) = \frac{1}{4} \left[ S_g(\omega + \omega_0) + S_g(\omega - \omega_0) \right] \quad 2.91 \]

Thus, modulation shifts the PSD of \( g(t) \) by \( \pm \omega_0 \). The power of \( \varphi(t) \) is half the power of \( g(t) \), that is,

\[ P_\varphi = \frac{1}{2} P_g \quad \omega_0 \geq 2\pi B \quad 2.92 \]