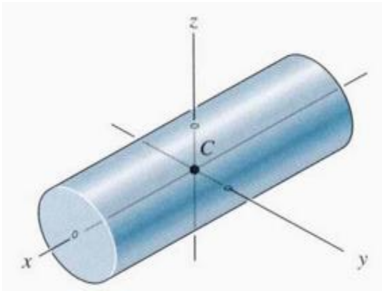
Center of Mass (CM):

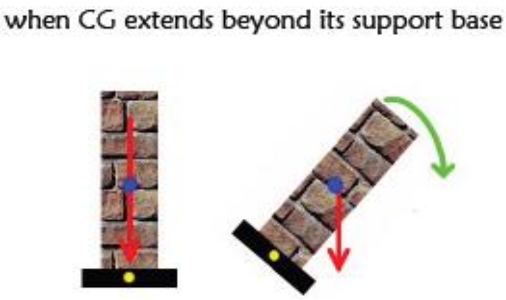


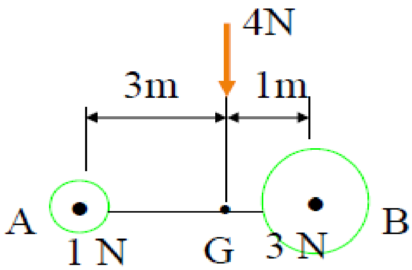
The center of mass is a point which locates the resultant mass of a system of particles or body.

It can be within the object (like a human standing straight) or outside the object like an object of an arc shape.

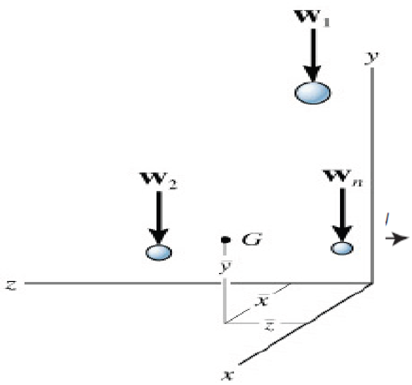
Center of Gravity:

Similarly, the center of gravity (CG) is a point which locates the resultant weight of a system of particles or body. The sum of moments due to the individual particles weights about center of gravity is equal to zero.





Center of Mass and gravity of a System of Particles:



Consider a system of n particles as shown in the figure. The net or the resultant weight is given as $W_R = \Sigma W$.

Summing the moments about the y-axis, we get:

$$\bar{x}\, W_R = x_1W_1 + x_2W_2 + \ldots.. + x_n W_n$$

where: x_1 represents x coordinate of W_1, etc..

W: weight of practical

W_R: resultant weight (Total weight)

Similarly, we can sum moments about the x- and z-axes to find the coordinates of G.

By replacing the (W) with a (M) in these equations, the coordinates of the center of mass can be found as the following:

$$\bar{x} = \frac{\sum m_i x_i}{M} \qquad \bar{y} = \frac{\sum m_i y_i}{M}$$

$$\bar{z} = \frac{\sum m_i z_i}{M}$$

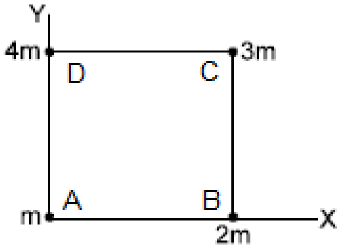
m_i: is the mass of each particle

M: is the sum of the masses of all particles

$\bar{x}_i$, $\bar{y}_i$, $\bar{z}_i$: is the position of each particle with respect to the origin

Example:

corners of a square of side a. Locate the centre of mass.



Solution:

Take the axes as shown in figure. The coordinates of the four particles are as follows.

Particle	Mass	x-coordinate	y-coordinate
A	m	0	0
B	2m	a	0
C	3m	a	a
D	4m	0	a

Hence, the coordinates of the centre of mass of the four particle system are

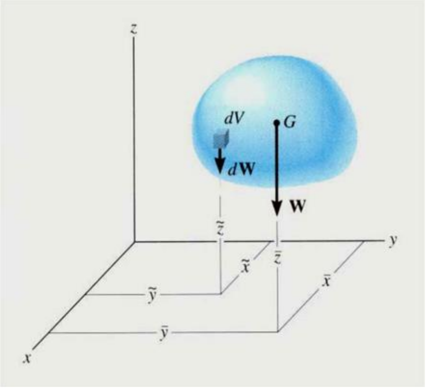
$$X = \frac{m.0 + 2\,m\,a + 3ma + 4m\,.\,0}{m + 2m + 3m + 4m} = \frac{a}{2}$$

$$Y = \frac{m.0 + 2\,m\,.0 + 3ma + 4m\,a}{m + 2m + 3m + 4m} = \frac{7a}{10}$$

The centre of mass is at $\left(\frac{a}{2}, \frac{7a}{10}\right)$

Center of Mass and Gravity of an Object

A rigid body can be considered as made up of an infinite number of particles. Hence, using the same principles as in the previous section, we get the coordinates of G by simply replacing the discrete summation sign (Σ) by the continuous summation sign (∫) and M by *dm*.



$$\bar{x} = \frac{\int x dW}{\int dW}; \qquad \bar{y} = \frac{\int y dW}{\int dW}; \qquad \bar{z} = \frac{\int z dW}{\int dW}$$

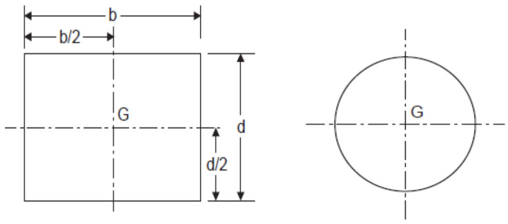
The center of mass of a rigid body can be determined using the same principles employed to determine the center of gravity. Therefore, the center of mass of a body can be expressed as;

$$\bar{x} = \frac{1}{M}\int x\, dm \quad , \qquad \bar{y} = \frac{1}{M}\int y\, dm$$

$$\bar{z} = \frac{1}{M}\int z\, dm$$

Centroid:

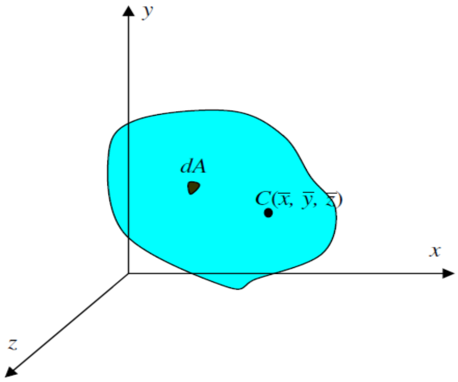
The centroid *C* is a point which defines the *geometric center* of an object. The centroid coincides with the center of mass or the center of gravity only if the material of the body is homogenous (density or specific weight is constant throughout the body). If an object has an axis of symmetry, then the centroid of object lies on that axis.



Its location with respect to the origin ($\bar{x}$, $\bar{y}$ and $\bar{z}$) can be determined using the same principles employed to determine the center of gravity of a body. In the case where the material composing a body is uniform or *homogeneous*, the density or specific weight will be constant throughout the body. These values will be factored out from the integrals in finding the center of mass and center of gravity and simplifying the expressions. In this specific case, the centers of mass, gravity and geometry coincide.

$$\bar{x} = \frac{\sum A_i x_i}{A}, \quad \bar{y} = \frac{\sum A_i y_i}{A}$$

i.e. $$\bar{x} = \frac{\int x dA}{A} \qquad \bar{y} = \frac{\int y dA}{A}$$



an area.

Note: If the material composing a body is uniform or homogeneous, the density or specific weight will be constant throughout the body, then the centroid is the same as the center of gravity or center of mass

Example:

Locate the centroid of the T-section shown in the Fig.

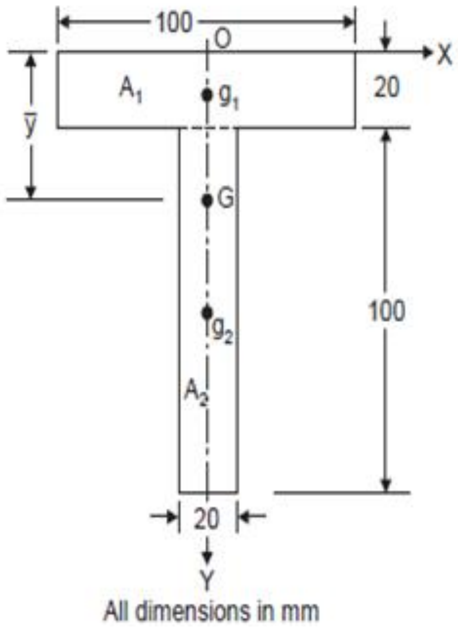
Solution. Selecting the axis as shown in Fig. 2.29, we can say due to symmetry centroid lies on *y* axis, *i.e.* $\bar{x} = 0$. Now the given T-section may be divided into two rectangles A_1 and A_2 each of size 100 × 20 and 20 × 100. The centroid of A_1 and A_2 are $g_1(0, 10)$ and $g_2(0, 70)$ respectively.

∴ The distance of centroid from top is given by:

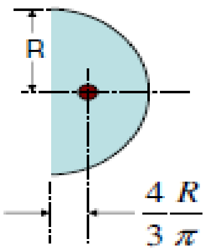
$$\bar{y} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100}$$

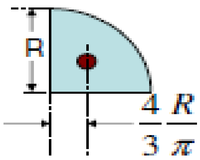
$= 40$ mm

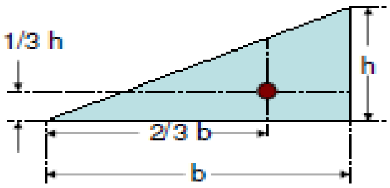
Hence, *centroid of T-section is on the symmetric axis at a distance 40 mm from the top.* **Ans.**



Centroid of common areas:



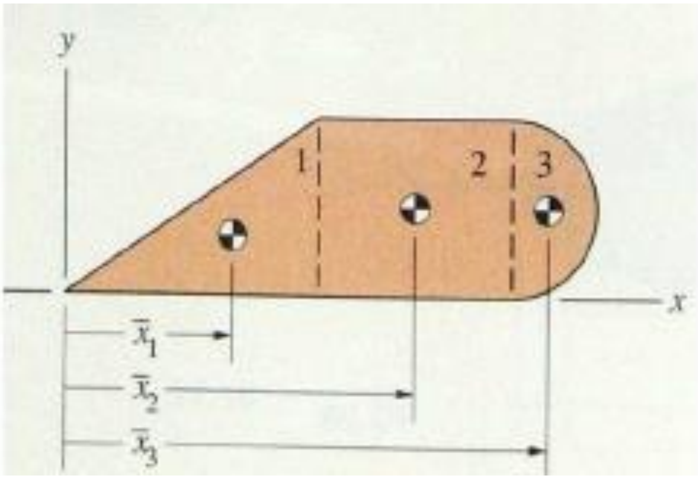




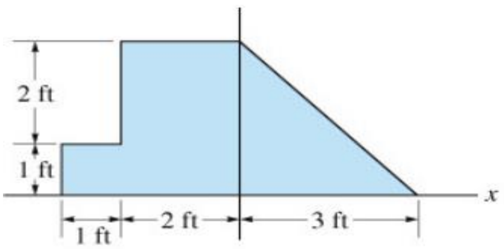
Centroid of composite area:

We can break this figure up into a series of shapes and find the location of the local centroid of each as shown in the following;

$$\bar{x} = \frac{\int_{A_1} x\,dA + \int_{A_2} x\,dA + \int_{A_3} x\,dA}{\int_{A_1} dA + \int_{A_2} dA + \int_{A_3} dA}$$



Example:



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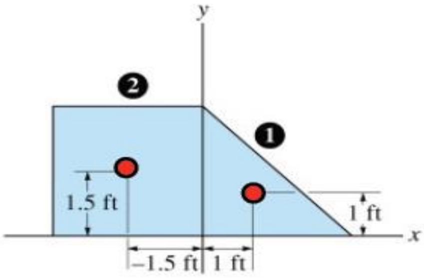

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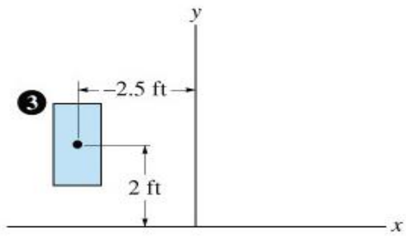
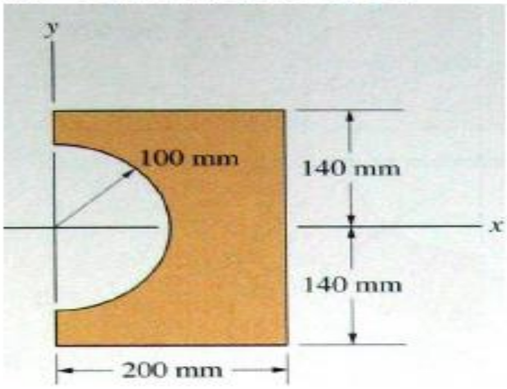


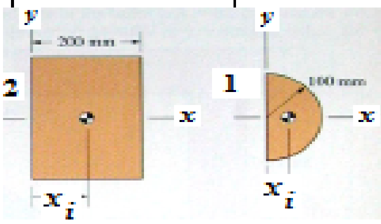
Example:

Determine the centroid of the area



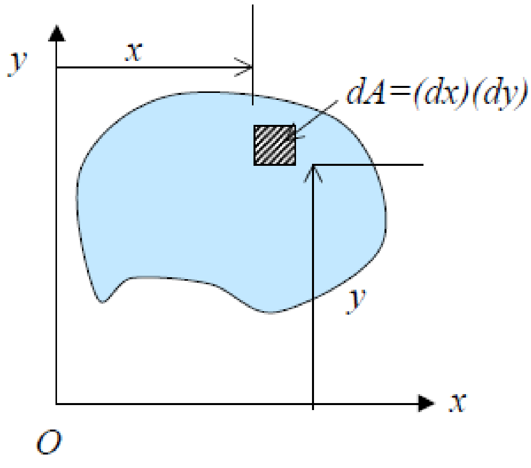
Solution:

	$\bar{x}_i$	A_i	$\bar{x}_i A_i$
Rectangle	100	(200)(280)	(100)[(200)(280)]
Semi-circle	$\frac{4(100)}{3\pi}$	$-\frac{1}{2}\pi(100)^2$	$-\frac{4(100)}{3\pi}\left[\frac{1}{2}\pi(100)^2\right]$



$$\bar{x} = \frac{x_1A_1 + x_2A_2}{A_1 + A_2} = \frac{(100)[(200)(280)] - \frac{4(100)}{3\pi}\left[\frac{1}{2}\pi(100)^2\right]}{(200)(280) - \frac{1}{2}\pi(100)^2} = 122mm$$

Moment of Inertia:



The moment of inertia can be considered as a shape factor which indicated how the material is distributed about the center of gravity of the cross-section. It's also defined as the capacity of a cross-section to resist bending. It is usually quantified in m4 or kgm2

So, the moment of inertia has a significant effect on the structural behavior of construction elements. The formulas used for determining the moment of inertia are:

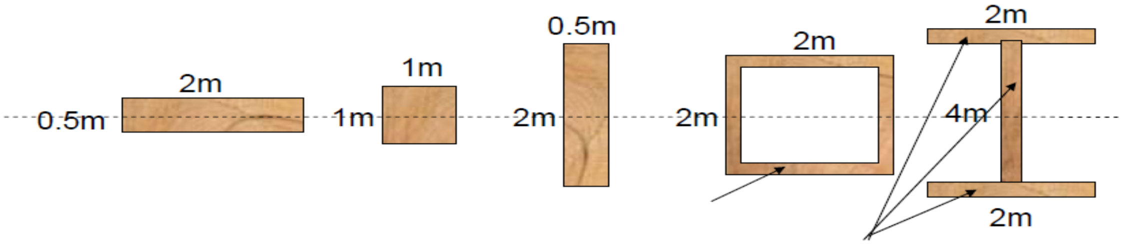
$$I_x = \Sigma y^2 dA$$
$$I_y = \Sigma x^2 dA$$

or by using calculus

$$I_x = \int y^2 dA$$
$$I_y = \int x^2 dA$$

$I\,x$: Moment of inertia about x axis
$I\,y$: Moment of inertia about y axis

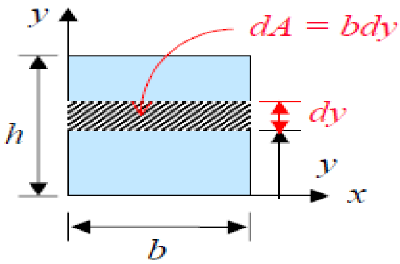
The physical meaning of moment of inertia:



Area	$1m^2$	$1m^2$	$1m^2$	$1m^2$	$1m^2$
Moment of Inertia	$0.02\ m^4$	$0.08\ m^4$	$0.33 m^4$	$0.55 m^4$	$2.79 m^4$

<u>Example:</u>

- **Moment of Inertia of a Rectangular Area.**

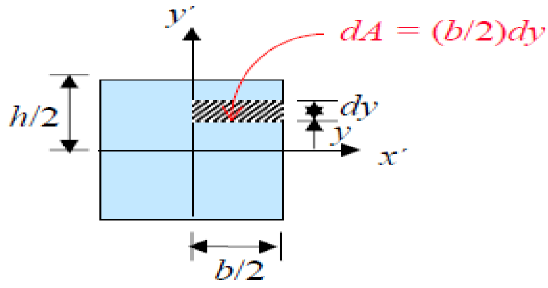


$$I_x = \int_A y^2 dA$$

$$= \int_0^h y^2 (bdy)$$

$$= \left.\frac{(by^3)}{3}\right|_0^h$$

$$= \frac{bh^3}{3}$$

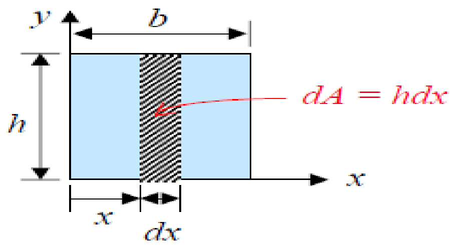


$$\bar{I}_x = I_{x'} = \int_A y^2 dA$$

$$= 4\int_0^{h/2} y^2 (\frac{b}{2} dy)$$

$$= 4(\frac{b}{2}) \left.\frac{y^3}{3}\right|_0^{h/2}$$

$$= \frac{bh^3}{12}$$

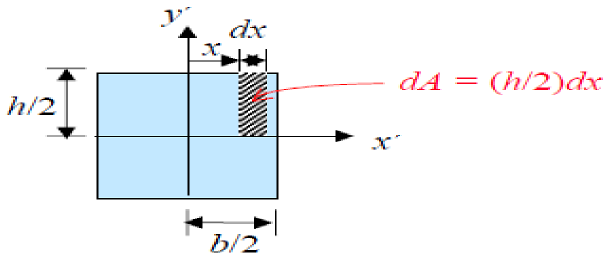


$$I_y = \int_A x^2 dA$$

$$= \int_0^b x^2 (hdx)$$

$$= \left.\frac{(hx^3)}{3}\right|_0^b$$

$$= \frac{hb^3}{3}$$



$$\bar{I}_y = I_{y'} = \int_A x^2 dA$$

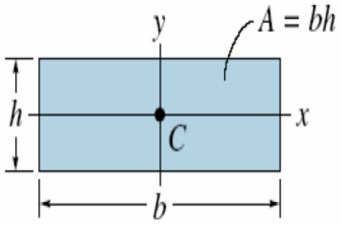
$$= 4\int_0^h x^2 (\frac{h}{2} dx)$$

$$= 4(\frac{h}{2}) \left.\frac{x^3}{3}\right|_0^{b/2}$$

$$= \frac{hb^3}{12}$$

Moment of inertia of common area:

Centroid Location | Area Moment of Inertia

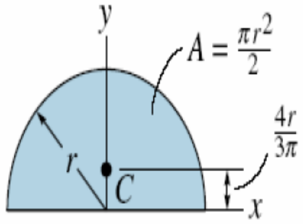


$$I_x = \frac{1}{12}bh^3$$

$$I_y = \frac{1}{12}hb^3$$

Rectangular area

Centroid Location | Area Moment of Inertia

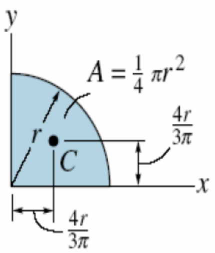


$$I_x = \frac{1}{8}\pi r^4$$

$$I_y = \frac{1}{8}\pi r^4$$

Semicircular area

Centroid Location | Area Moment of Inertia

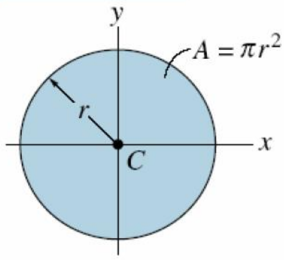


$$I_x = \frac{1}{16}\pi r^4$$

$$I_y = \frac{1}{16}\pi r^4$$

Quarter circle area

Centroid Location | Area Moment of Inertia

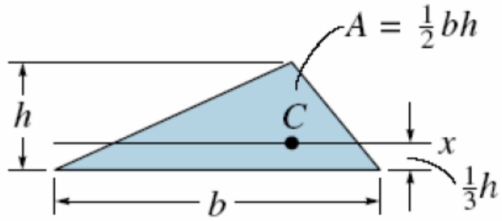


$$I_x = \frac{1}{4}\pi r^4$$

$$I_y = \frac{1}{4}\pi r^4$$

Circular area

Centroid Location | Area Moment of Inertia



$$I_x = \frac{1}{36}bh^3$$

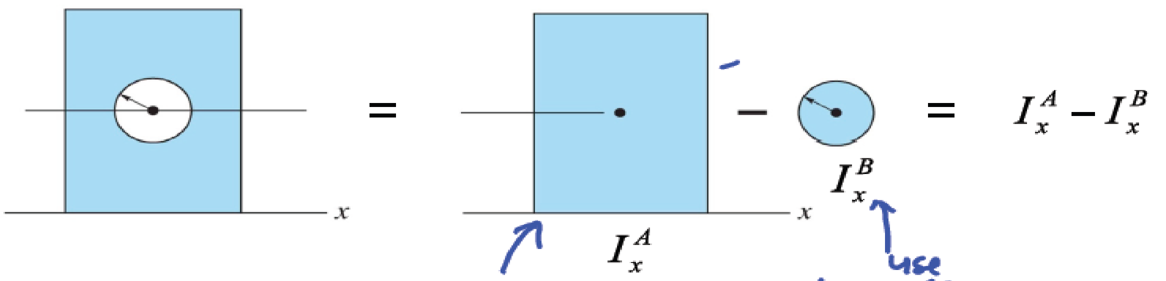
Triangular area

Moment of inertia of composite areas:

Moment of inertia of a complex area

Break into simpler shapes, then compute moment of inertia of them (using the table)

Add (or subtract) m.o.i. of simple shapes



use eq. to Compute I_x^A

use eq.

Parallel Axis Theorem:

If you know the moment of inertia about a **centroidal axis** of a figure, you can calculate the moment of inertia about any **parallel axis to the centroidal axis** using a simple formula;

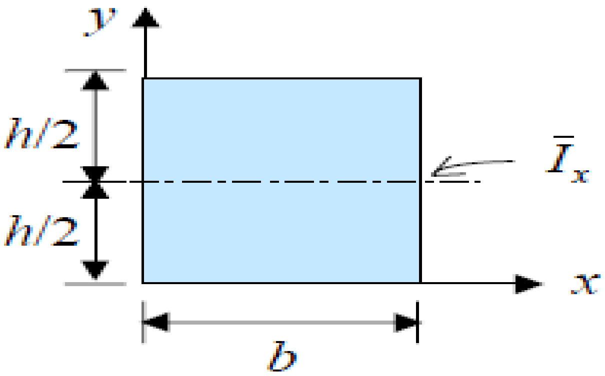
$$I_z = I_{\bar{z}} + Ay^2$$

I_z : moment of inertia around z axis, $I_{\bar{z}}$: moment of inertia around z axis

A : area of the figure

y : distance from the centroid to ***z*** axis along y axis

Example: find the moment of inertia using parallel axis theorem



$$\bar{I}_x = \frac{bh^3}{12}$$

$$I_x = \frac{bh^3}{3}$$

$$I_x = \bar{I}_x + Ad^2$$

$$= \frac{bh^3}{12} + (bh)(\frac{h}{2})^2$$

$$= \frac{bh^3}{12} + \frac{bh^3}{4}$$

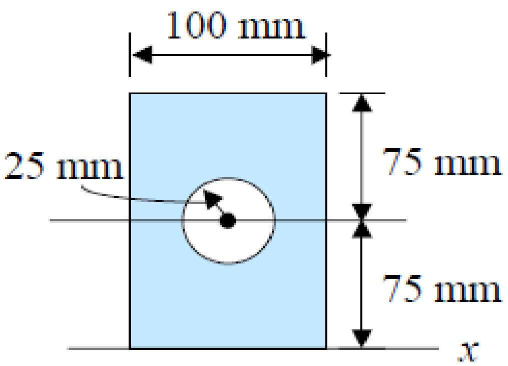
$$I_x = \frac{bh^3}{3}$$

Where d: is the distance from the centroid to (x) axis

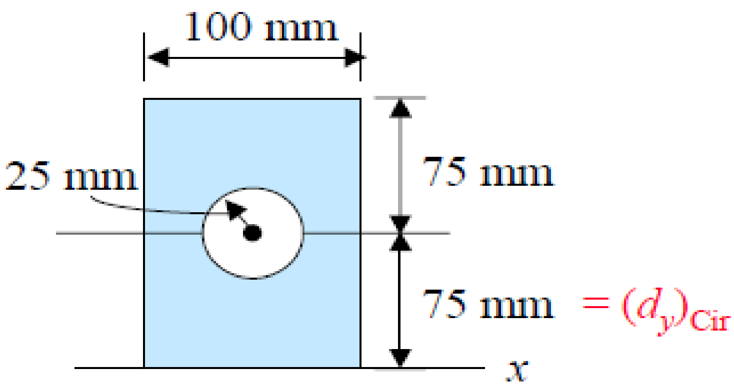
A : area of the triangle

Example:

Compute the moment of inertia of the composite area shown.



Solution:



$$I_x = \left(\frac{bh^3}{3}\right)_{Rect} - \left(\bar{I}_x + A d_y^{\,2}\right)_{Cir}$$

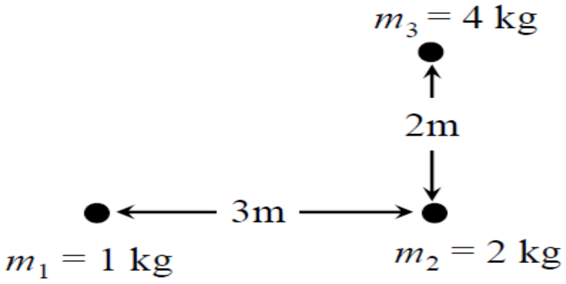
$$= \left[\frac{1}{3}(100)(150)^3\right]_{Rect} - \left[\frac{1}{4}\pi(25)^4 + (\pi \times 25^2)(75)^2\right]_{Cir}$$

$$= 101 \text{x} 10^6 \text{ mm}^4$$

Examples & solutions

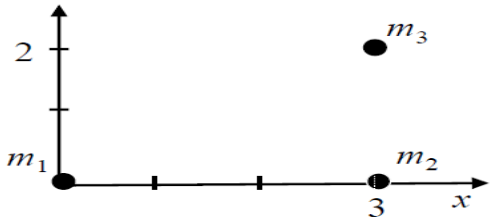
Example

Find the centre of mass of the distribution of particles shown:



Solution:

Choose a coordinate axes (any will do)

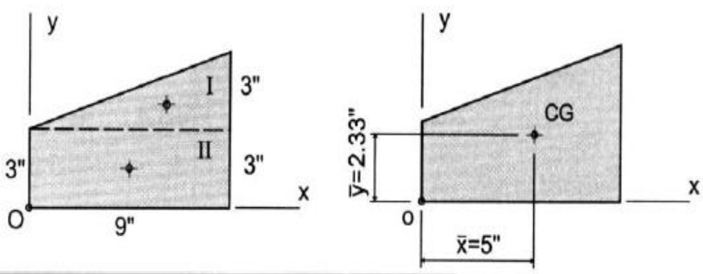


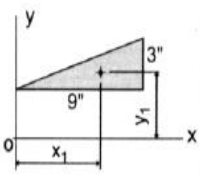
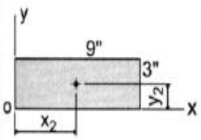
$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{(1)(0) + (2)(3) + (4)(3)}{1+2+4} = \frac{18}{7} = 2.57$$

$$y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{(1)(0) + (2)(0) + (4)(2)}{1+2+4} = \frac{8}{7} = 1.14$$

Example

Determine the centroidal x and y distances for the composite area shown. Use the lower left corner of the trapezoid as the reference origin.

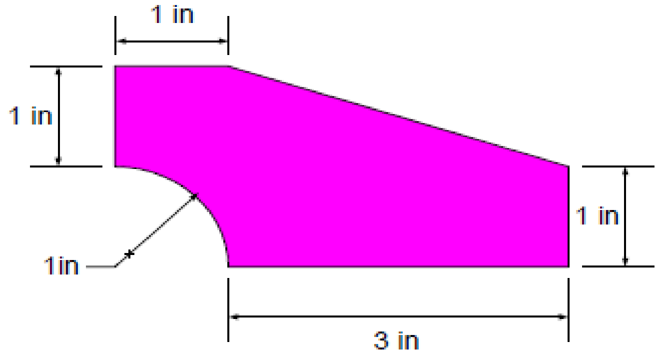


Component	Area (A) (in.^{2})	$\bar{x}$ (in.)	$\bar{x}A$ (in.^{3})	$\bar{y}$ (in.)	$\bar{y}A$ (in.^{3})
 (a)	$\frac{9''(3'')}{2} = 13.5$ in.^{2}	6"	81 in.^{3}	4"	54 in.^{3}
 (b)	9" (3") = 27 in.^{2}	4.5"	121.5 in.^{3}	1.5"	40.5 in.^{3}
	$A = \sum A = 40.5 in.^2$		$\sum \bar{x}A = 202.5 in.^3$		$\sum \bar{y}A = 94.5 in.^3$

$$\hat{x} = \frac{202.5in^3}{40.5in^2} = 5in$$

$$\hat{y} = \frac{94.5in^3}{40.5in^2} = 2.33in$$

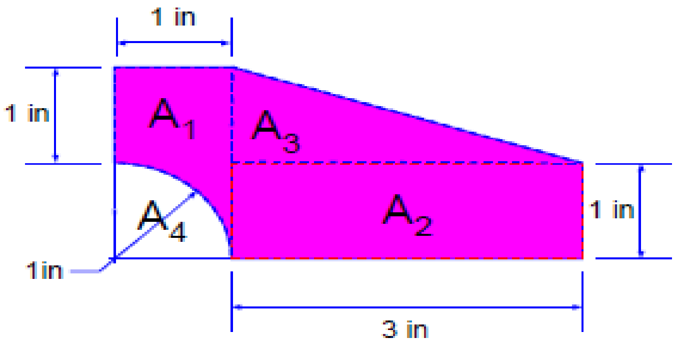
Example: Find the centroid of the following shape:



Solution:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i} \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i A_i}{\sum_{i=1}^{n} A_i}$$

We can break this figure up into a series of shapes and find the location of the local centroid of each

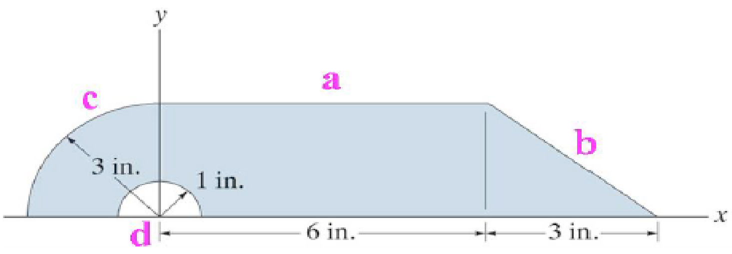


Make a table contains the parameters as shown;

ID	Area	x_i	x_i*Area	y_i	y_i*Area
	(in^2)	(in)	(in^3)	(in)	(in^3)
A_1	2	0.5	1	1	2
A_2	3	2.5	7.5	0.5	1.5
A_3	1.5	2	3	1.333333	2
A_4	-0.7854	0.42441	-0.33333	0.42441	-0.33333
	5.714602		11.16667		5.166667
	$\bar{x}$	1.9541	$\bar{y}$	0.904117	

$\bar{X} = 1.9541$, $\bar{y} = 0.904117$ Ans.

Example:



Given: The part shown.

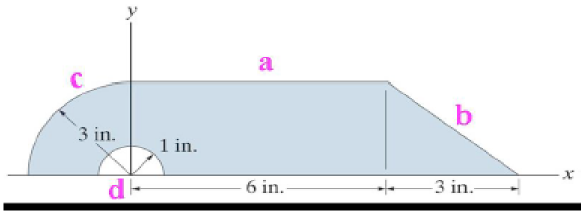
Find: The centroid of the part.

Plan: Follow the steps for analysis.

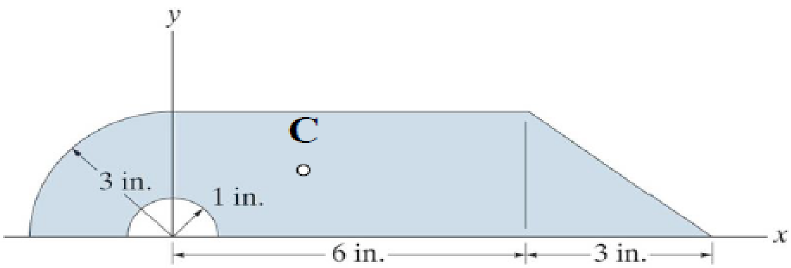
Solution:

1. This body can be divided into the following pieces: rectangle (a) + triangle (b) + quarter circular (c) – semicircular area (d). Note the negative sign on the hole!

Steps 2 & 3: Make up and fill the table using parts a, b, c, and d.



Segment	Area A (in^2)	$\tilde{x}$ (in)	$\tilde{y}$ (in)	$\tilde{x}$ A (in^3)	$\tilde{y}$ A (in^3)
Rectangle	18	3	1.5	54	27
Triangle	4.5	7	1	31.5	4.5
Q. Circle	$9\pi / 4$	$-4(3)/(3\pi)$	$4(3)/(3\pi)$	-9	9
Semi-Circle	$-\pi / 2$	0	$4(1)/(3\pi)$	0	- 2/3
Σ	28.0			76.5	39.83



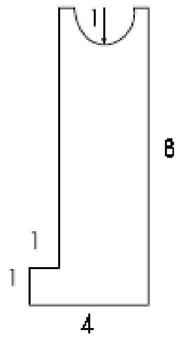
4. Now use the table data results and the formulas to find the coordinates of the centroid.

Area A	$\tilde{x}$ A	$\tilde{y}$ A
28.0	76.5	39.83

$$\bar{x} = (\Sigma \tilde{x} A) / (\Sigma A) = 76.5 \text{ in}^3 / 28.0 \text{ in}^2 = 2.73 \text{ in}$$

$$\bar{y} = (\Sigma \tilde{y} A) / (\Sigma A) = 39.83 \text{ in}^3 / 28.0 \text{ in}^2 = 1.42 \text{ in}$$

Example: Centroid of an area



The area can be considered a 1×1 square plus a 4×8 rectangle minus a semicircle with radius $a = 1$.

From the table in the NCEES Handbook, y_c for a square or rectangle is one-half the height.

For a semicircle, the distance from the base to the centroid is:

$4a/(3\pi) = 4(1)/(3\pi) = 0.424$

The base of the semicircle is 8 units above the x-axis, so for the semicircle:

$y_c = 8 - 0.424 = 7.576$

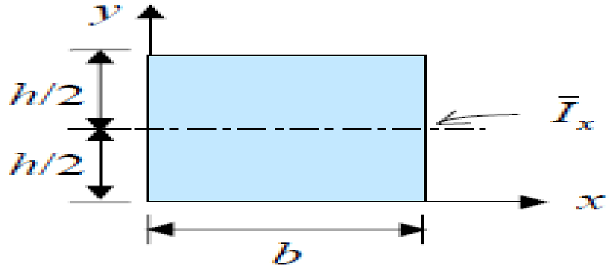
The area of the semicircle is $\pi a^2/2 = 1.57$

Since the semicircle is a negative area, it gets a negative sign in the summation.

The total y_c is the sum of the individual areas times the individual y_c divided by the total area.

$$y_{ac} = \frac{\sum A_i y_i}{\sum A_i} = \frac{(1.0)(0.5) + (32)(4.0) + (-1.57)(7.576)}{1.0 + 32.0 - 1.57} = 4.98$$

Example: find the moment of inertia using parallel axis theorem



$$\bar{I}_x = \frac{bh^3}{12} \qquad I_x = \frac{bh^3}{3}$$

$$I_x = \bar{I}_x + Ad^2$$

$$= \frac{bh^3}{12} + (bh)\left(\frac{h}{2}\right)^2$$

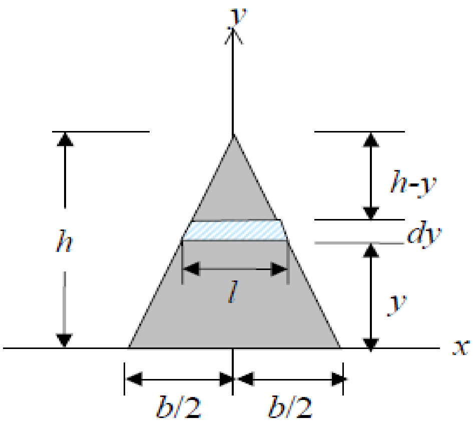
$$= \frac{bh^3}{12} + \frac{bh^3}{4}$$

$$I_x = \frac{bh^3}{3}$$

Where d: is the distance from the centroid to (x) axis

A : area of the triangle

Example: Find the moment of inertia of a triangle



$$dA = l\,dy$$

Using similar triangles, we have

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

Integrating from $y = 0$ to $y = h$, we obtain

$$I_x = \int y^2\, dA$$

$$= \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h (hy^2 - y^3)dy$$

$$= \frac{b}{h}[h\frac{y^3}{3} - \frac{y^4}{4}]_0^h = \frac{bh^3}{12}$$

$$I_x = \bar{I}_x + Ad^2$$

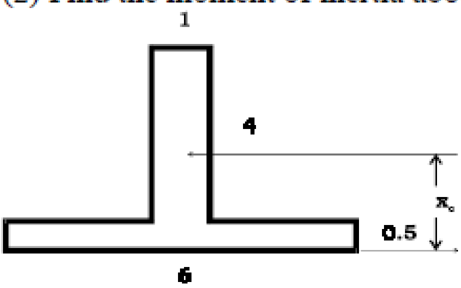
$$\bar{I}_x = I_x - Ad^2$$

$$= \frac{bh^3}{12} - (\frac{bh}{2})(\frac{h}{3})^2 = \frac{bh^3}{36}$$

Examples: Moment of Inertia

(1) Find the moment of inertia for the shape about the x-axis. (All dimensions show are in mm.)

(2) Find the moment of inertia about the horizontal centroid of the total shape.



(1) Consider the shape as two adjacent rectangles. The bottom rectangle has a moment of inertia taken about its edge (the x-axis) of I_x (lower)= $bh^3/3 = 0.25$ mm^4. The moment of inertia for the upper rectangle taken about the axis of its centroid is I_{xc} (upper)= $bh^3/12 = 5.33$ mm^4. Use the transfer theorem to move the upper rectangle to the x-axis.

A = (4 mm)(1 mm) = 4 mm^2
I_x (upper)= I_{xc} (upper) + Ad^2 = 5.33 mm^4 + (4 mm^2)(0.5 + (4mm/2))^2 = 30.33 mm^4
I_x total = I_x (lower) + I_x (upper) = 30.33 mm^4 + 0.25 mm^4 = 30.58 mm^4

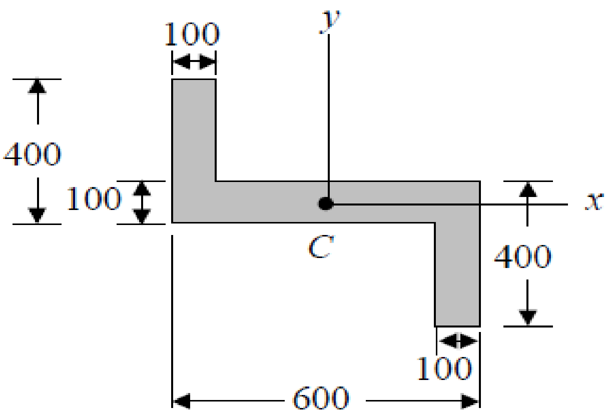
(2) The centroid of the total shape is:

$$y_c = \frac{\sum A_i y_i}{\sum A_i} = \frac{(4.0)(2.5) + (3.0)(0.25)}{4.0 + 3.0} = 1.536 \text{ mm}$$

Use the transfer axis theorem to convert the x-axis to the centroid:
$I_x = I_{cx} + Ad^2$
$I_{cx} = I_x - Ad^2$ = 30.58 mm^4 - (4 mm^2 + 3 mm^2)(1.536 mm)^2 = 14.07 mm^4

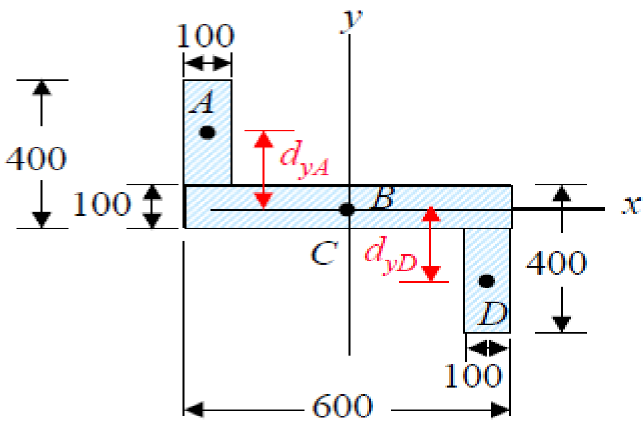
Example:

Determine the moments of inertia of the beam's cross-sectional area shown about the *x* and *y* centroidal axes.



Dimension in mm

Solution:



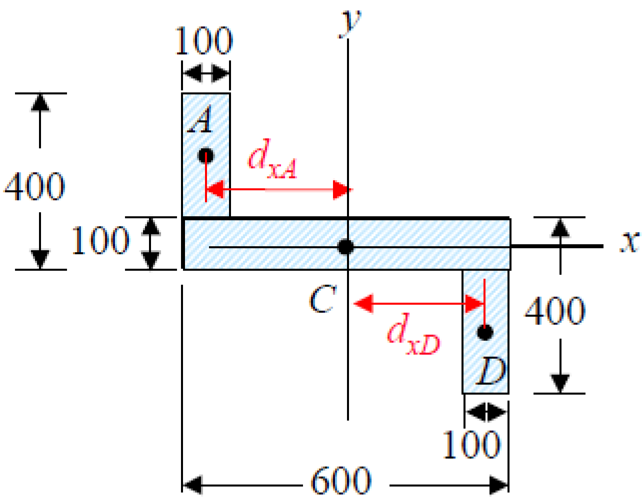
Dimension in mm

$$I_x = (\bar{I}_x + Ad_y^{\,2})_A + (\bar{I}_x + A\overset{0}{d_y^{\,2}})_B + (\bar{I}_x + Ad_y^{\,2})_C$$

$$= [\frac{1}{12}(100)(300)^3 + (100 \times 300)(200)^2] + [\frac{1}{12}(600)(100)^3 + 0]$$

$$+ [\frac{1}{12}(100)(300)^3 + (100 \times 300)(200)^2]$$

$$= 2.9\text{x}10^9 \text{ mm}^4$$



Dimension in mm

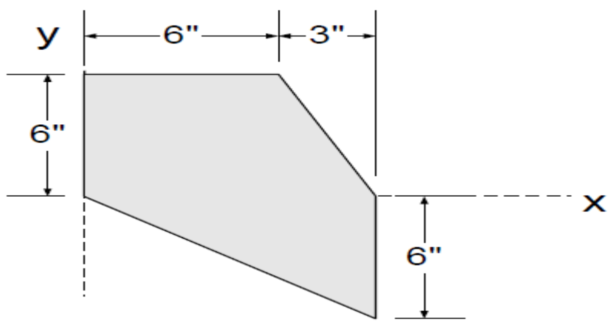
$$I_y = (\bar{I}_y + Ad_x{}^2)_A + (\bar{I}_y + \underbrace{Ad_x{}^2}_{0})_B + (\bar{I}_y + Ad_x{}^2)_C$$

$$= [\frac{1}{12}(300)(100)^3 + (100 \times 300)(250)^2]_A + [\frac{1}{12}(100)(600)^3 + 0]_B$$

$$+ [\frac{1}{12}(300)(100)^3 + (100 \times 300)(250)^2]_C$$

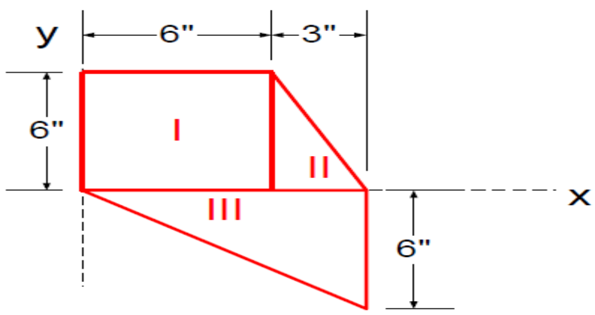
$$= 5.6\text{x}10^9 \text{ mm}^4$$

Example: Find the moment of inertia around *(y)* and *(x)* axis



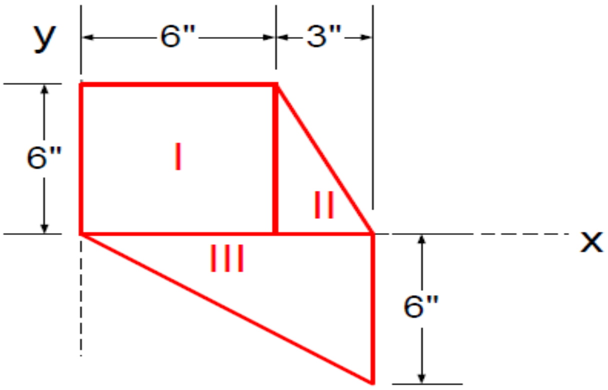
Solution:

We can divide up the area into smaller areas with shapes from the table



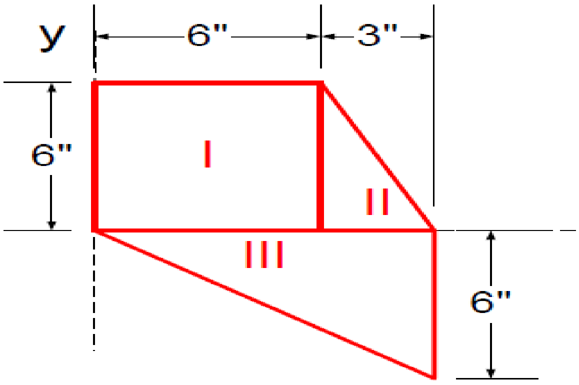
Since the parallel axis theorem will require the area for each section, that is a reasonable place to start

ID	Area (in^2)
I	36
II	9
III	27



We can locate the centroid of each area with respect the y axis.

ID	Area (in^2)	$xbar_i$ (in)
I	36	3
II	9	7
III	27	6



From the table in the back of the book we find that the moment of inertia of a rectangle about its y-centroid axis is

$$I_{\bar{y}} = \frac{1}{12} b^3 h$$

In this example, for Area I, b=6" and h=6"

$$I_{\bar{y}} = \frac{1}{12}(6in)(6in)^3$$

$$I_{\bar{y}} = 108 in^4$$

moment of inertia of the II triangle

$$I_{\bar{y}} = \frac{1}{36} b h^3$$

$$I_{\bar{y}} = \frac{1}{36}(6in)(3in)^3$$

$$I_{\bar{y}} = 4.5 in^4$$

The same is true for the III triangle

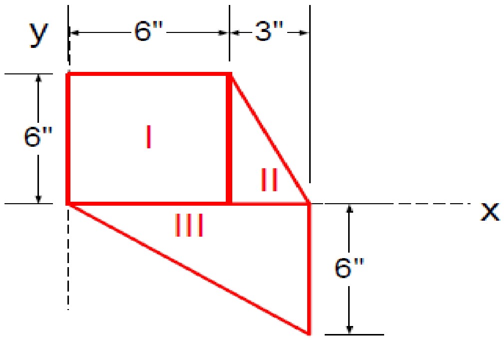
$$I_{\bar{y}} = \frac{1}{36} bh^3$$

$$I_{\bar{y}} = \frac{1}{36} (6in)(9in)^3$$

$$I_{\bar{y}} = 121.5 in^4$$

Now we can enter the I_{ybar} for each sub-area into the table

Sub-Area	Area (in^2)	$xbar_i$ (in)	I_{ybar} (in^4)
I	36	3	108
II	9	7	4.5
III	27	6	121.5



We can then sum the I_y and the $A(d_x)^2$ to get the moment of inertia for each sub-area

Sub-Area	Area (in^2)	$xbar_i$ (in)	I_{ybar} (in^4)	$A(d_x)^2$ (in^4)	$I_{ybar} + A(d_x)^2$ (in^4)
I	36	3	108	324	432
II	9	7	4.5	441	445.5
III	27	6	121.5	972	1093.5

Where A : area of the figure

dy : distance from the centroid to ***z*** axis along y axis

xbar, ybar means $\bar{x}$ and $\bar{y}$ alternatively

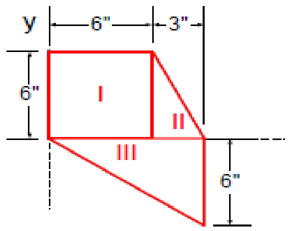
And if we sum that last column, we have the I_y for the composite figure

Sub-Area	Area (in^2)	$xbar_i$ (in)	I_{ybar} (in^4)	$A(d_x)^2$ (in^4)	$I_{ybar} + A(d_x)^2$ (in^4)
I	36	3	108	324	432
II	9	7	4.5	441	445.5
III	27	6	121.5	972	1093.5
					1971

$I_y = 1971\ in^4$ Ans.

We perform the same type of analysis for the I_x

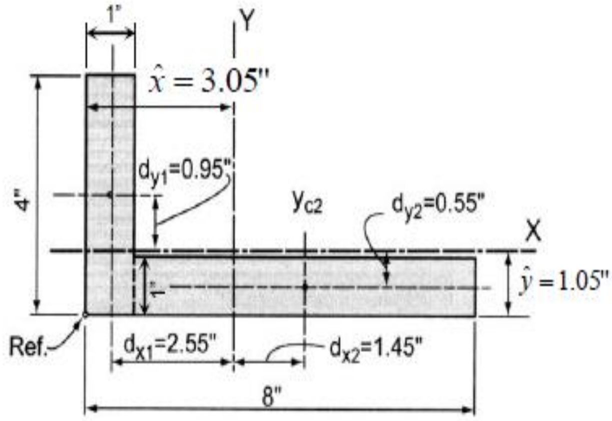
Sub-Area	Area (in^2)	$ybar_i$ (in)	I_{xbar} (in^4)	$A(d_y)^2$ (in^4)	$I_{xbar} + A(d_y)^2$ (in^4)
I	36	3	108	324	432
II	9	2	18	36	54
III	27	-2	54	108	162
					648

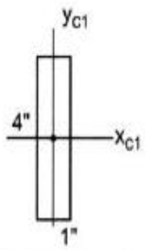
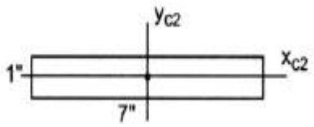


$I_x = 648\ in^4$ Ans.

Example:

Find the moments of inertia ($\hat{x} = 3.05$", $\hat{y} = 1.05$").



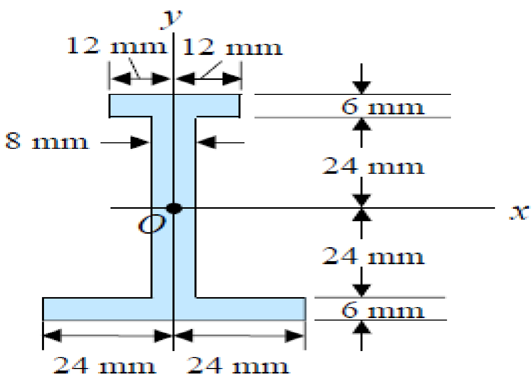
Component	I_{xc} (in.^{4})	d_y (in.)	Ad_y^2 (in.^{4})	I_{yc} (in.^{4})	d_x (in.)	Ad_x^2 (in.^{4})
	$\frac{(1)(4)^3}{12} = 5.33$	0.95	3.61	$\frac{(4)(1)^3}{12} = 0.33$	2.55	26.01
	$\frac{(7)(1)^3}{12} = 0.58$	0.55	2.12	$\frac{(1)(7)^3}{12} = 28.58$	1.45	14.72
	$\sum I_{xc} = 5.91$		$\sum Ad_y^2 = 5.73$	$\sum I_{yc} = 28.91$		$\sum Ad_x^2 = 40.73$

I_x= 5.91 + 5.73 = 11.64 in^4 Ans.

I_y= 28.91 + 40.73 = 69.64 in^4 Ans.

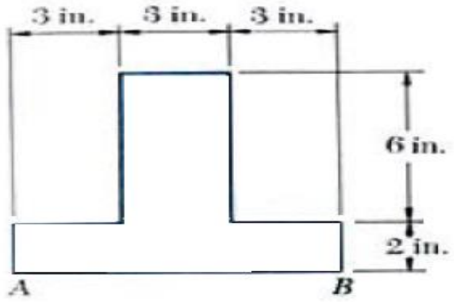
Question:

Determine the moments of inertia of the shaded area with respect to the x and y axes.



Example: Find the Moment of Inertia of this object around *AB*

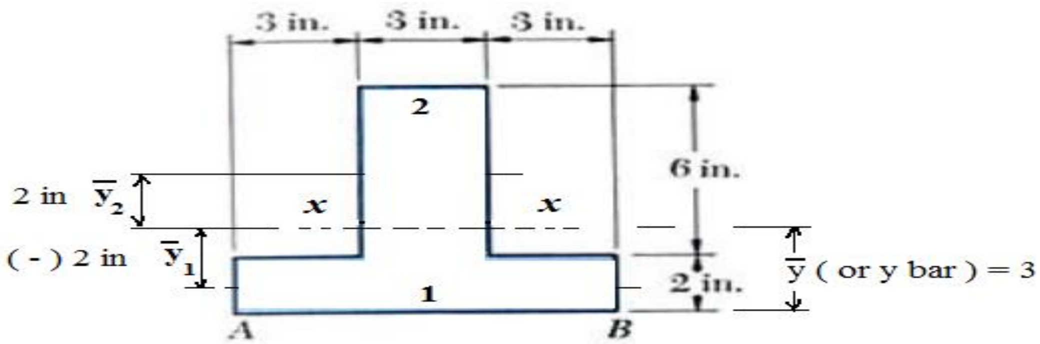
First we divide the object into two standard shapes present in the reference tables, the find the MI for each respective shape.



Set up the reference axis at AB and find the centroid

Bodies	A_i	y_i	y_i*A_i	I_i	$d_i=y_i$-ybar	$d_i^2A_i$
1	18	1	18			
2	18	5	90			
	36		108			

$$\bar{y} = \frac{\sum y_i A_i}{\sum A_i} = \frac{108 \text{ in}^3}{36 \text{ in}^2} = 3.0 \text{ in.}$$



Find the moment of inertia from the table:

Bodies	A_i	y_i	y_i*A_i	I_i	$d_i=y_i$-ybar	$d_i^2A_i$
1	18	1	18	6	-2	72
2	18	5	90	54	2	72
	36		108	60		144
ybar	3	in.				
I	204	in^4				

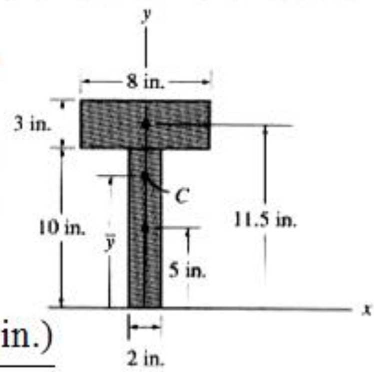
$$I_x = \sum \overline{I}_{xi} + \sum (y_i - \bar{y})^2 A_i$$

$$= 60 \text{ in}^4 + 144 \text{ in}^4 = 204 \text{ in}^4$$

Example:

Locate the centroid C of the cross-sectional area for the T-beam

Solution I



The y axis is placed along the axis of symmetry so that $\bar{x}=0$ To obtain $\bar{y}$ we will establish the x axis (reference axis) thought the base of the area.l The area is segmented into two rectangles and the centroidal location for each is established.

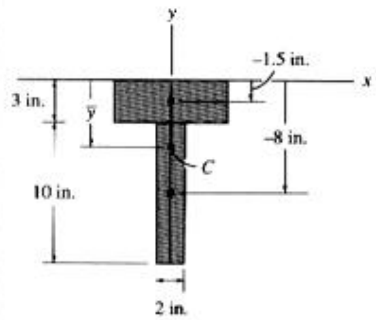
$$\bar{y}=\frac{\sum \tilde{y}A}{\sum A}=\frac{[5\text{in.}](10\text{ in.})(2\text{ in.})+[11.5\text{ in}](3\text{ in.})(8\text{ in.})}{(10\text{ in.})(2\text{ in.})+(3\text{ in.})(8\text{ in.})}$$
$$=8.55\text{in.}$$

Solution II

Using the same two segments, the x axis can be located at the top of the area. Here

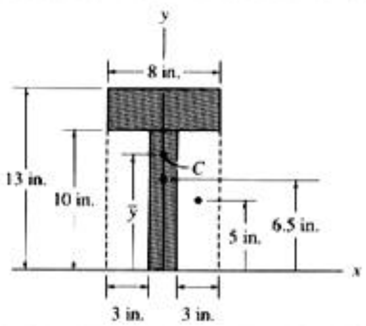
$$\bar{y}=\frac{\sum \tilde{y}A}{\sum A}=\frac{[-1.5\text{in.}](3\text{ in.})(8\text{ in.})+[-8\text{ in}](10\text{ in.})(2\text{ in.})}{(3\text{ in.})(8\text{ in.})+(10\text{ in.})(2\text{ in.})}$$
$$=-4.45\text{in.}$$

The negative sign indicates that C is located *below* the origin, which is to be expected. Also note that from the two answers 8.55 in + 4.45 in =13.0 in., which is the depth of the beam as expected



Solution III

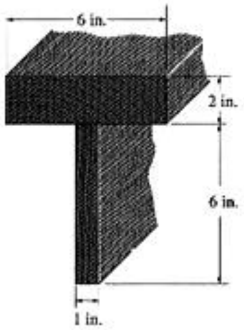
It is also possible to consider the cross-sectional area to be one large rectangle *less* two small rectangles. Hence we have



$$\bar{y}=\frac{\sum \tilde{y}A}{\sum A}=\frac{[6.5\text{in.}](13\text{ in.})(8\text{ in.})-2\,[5\text{ in}](10\text{ in.})(3\text{ in.})}{(13\text{ in.})(8\text{ in.})-2(10\text{ in.})(3\text{ in.})}$$
$$=8.55\text{in.}$$

Example:

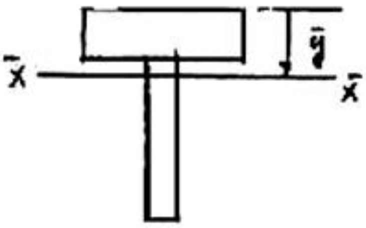
Determine the moment of inertia of the shaded area with respect to a horizontal axis passing through the centroid of the section



Solution:

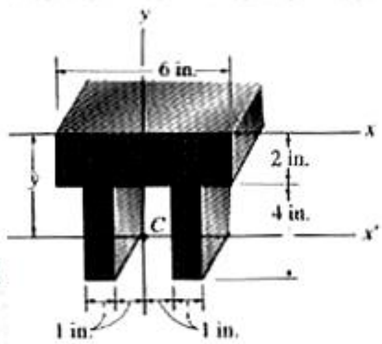
$$\bar{y}=\frac{\sum \tilde{y}A}{\sum A}=\frac{1(2)(6)+5(6)(1)}{2(6)+1(6)}=2.333\,\text{in}$$

$$I_{\bar{x}\bar{x}}=\frac{1}{12}(6)(2)^3+2(6)(2.333-1)^2+\frac{1}{12}(1)(6)^3+1(6)(5-2.333)^2=86\,\text{in}^4$$



Example:

Determine the moment of inertia of the beam's cross-sectional area about the y axis

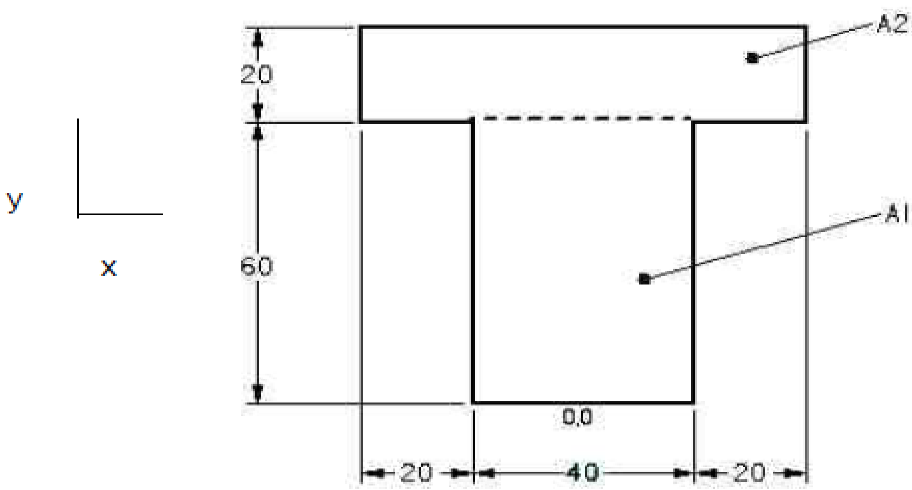


Solution:

$$I_y=\frac{1}{12}(2)(6)^3+2\left[\frac{1}{12}(4)(1)^3+1(4)(1.5)^2\right]=54.7\,\text{in}^4$$

Example:

Determine the moment of inertia for Figure 1 relative to the x-axis



Solution:

Step 1 – Determine the moment of inertia for area 1

$$
\begin{aligned}
I_{x,1} &= \bar{I}_{x',1} + A_1 d_1^2 \\
&= \frac{1}{12} b_1 h_1^{\,3} + b_1 \cdot h_1 \cdot (\bar{y}_1 - \bar{Y})^2 \\
&= \frac{1}{12} 40mm \cdot (60mm)^3 + 2400mm^2 \cdot (30mm - 46mm)^2 \\
&= 1.33 \cdot 10^6 mm^4
\end{aligned}
$$

Step 2 – Determine the moment of inertia for area 2

$$
\begin{aligned}
I_{x,2} &= \bar{I}_{x',2} + A_2 d_2^2 \\
&= \frac{1}{12} b_2 h_2^{\,3} + b_2 \cdot h_2 \cdot (\bar{y}_2 - \bar{Y})^2 \\
&= \frac{1}{12} 80mm \cdot (20mm)^3 + 1600mm^2 \cdot (70mm - 46mm)^2 \\
&= 9.75 \cdot 10^5 mm^4
\end{aligned}
$$

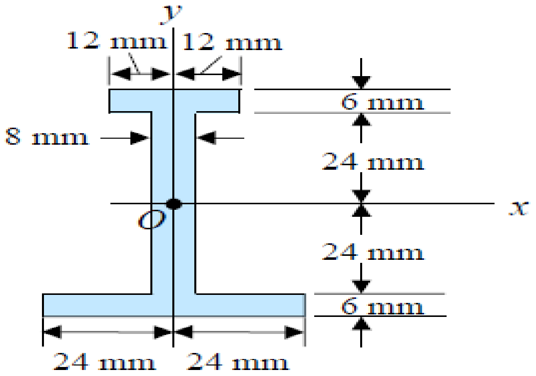
Step 3 - Find the total moment of inertia about the x axis

$$
\begin{aligned}
I_{x,T} &= I_{x,1} + I_{x,2} \\
&= 2.3 \cdot 10^6 mm^4 \\
&= 2.3 \cdot 10^6 mm^4 \cdot (1.0 \cdot 10^{-3} m/mm)^4 \\
&= 2.3 \cdot 10^{-6} m^4
\end{aligned}
$$

Determine the moment of inertia of the following shapes with respect to the x and y axes

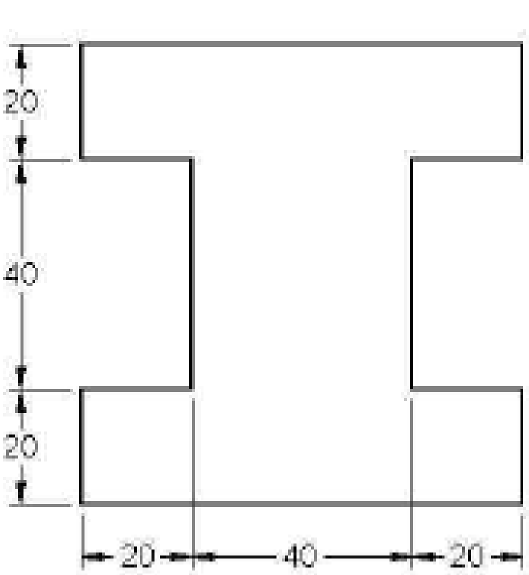
Question:

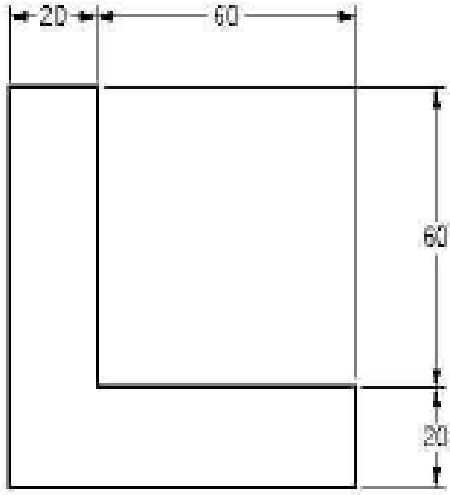
Determine the moments of inertia of the shaded area with respect to the x and y axes.

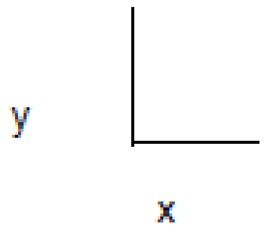


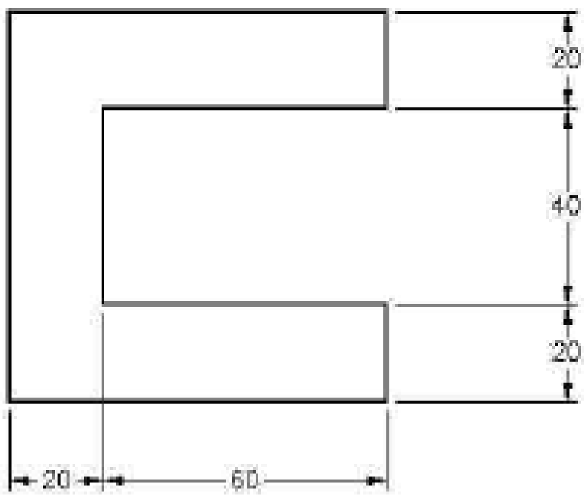
Question:

Determine the moment of inertia of the following shapes with respect to the x and y axes



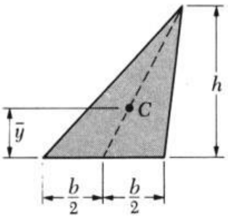
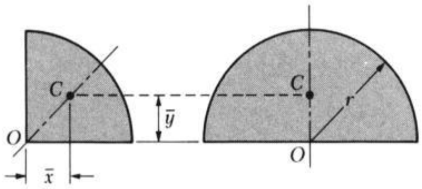
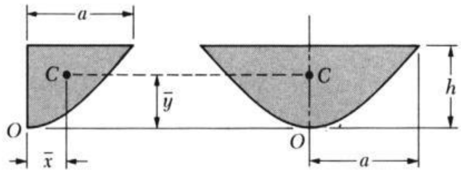
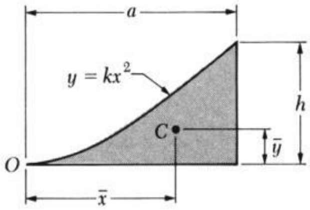
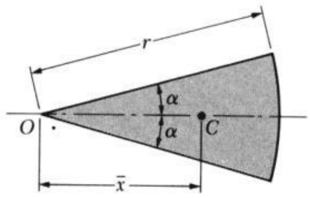
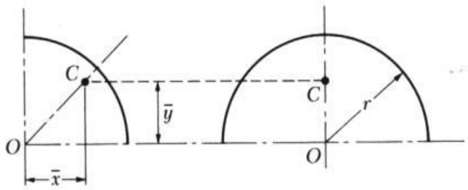
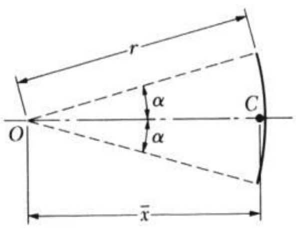






Note: all dimensions are in mm

Centroids of Common Shapes of Areas and Lines

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$