# Fundamentals of Communications Engineering

**Department of Communications Engineering, College of Engineering, University of Diyala, 2016-2017**

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# Fourier Transform Worked Examples

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**EX. 1** Find the Fourier transform of $x(t-3)$, if you know

$$X(f) = \frac{5(3+f)}{2f^2 - 3f + j4f}$$

**Solu.** if $x(t) \xrightarrow{F.T} X(f)$

$$x(t-3) \xrightarrow{F.T.} X(f)\, e^{-j2\pi f 3}$$

$\therefore$ $X(f)$ for our problem is

$$X(f) = \frac{5(3+f)}{2f^2 - 3f + j4f}\, e^{-j\omega 3}$$

**EX. 2** you have $X(f) = 4\,\text{sinc}^2(f) - 8\sin^2(f)$. Find the Fourier transform of $x(4t)$.

**solution** The time scaling property states that

$$x(\alpha t) \longrightarrow \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$$

$$\therefore X(f) = \frac{1}{4}\left[4\,\text{sinc}^2\left(\frac{f}{4}\right) - 8\sin^2\left(\frac{f}{4}\right)\right]$$

$$= \text{sinc}^2\left(\frac{f}{4}\right) - 2\sin^2\left(\frac{f}{4}\right)$$

EX.3 Determine the Fourier transform of $e^{-j8\pi t} x(t)$, where $X(f) = rect(f) + sinc^2(3f)$.

Solution frequency translation property shows:

$$x(t)\, e^{-j\omega_0 t} \xrightarrow{F.T.} X(f + f_0) \qquad \omega_0 = 2\pi f_0 \to f_0 = 4$$

$$\therefore X(f) = rect(f+4) + sinc^2(3(f+4))$$

EX.4 what is the Fourier transform of $x(-9t)$ if you have given $X(f) = 81\, rect^2(f) - 18\, sinc(27f)$?

Solution According to time scaling property

$$x(\alpha t) \xrightarrow{F.T.} \frac{1}{|\alpha|} X(f)$$

$$\therefore X(f) = \frac{1}{|-9|}\left[81\, rect^2(f/(-9)) - 18\, sinc\left(\frac{27f}{-9}\right)\right]$$

$$= \frac{81}{9} rect^2\left(\frac{f}{-9}\right) - \frac{18}{9} sinc\left(\frac{27f}{-9}\right)$$

$$= 9\, rect^2\left(\frac{f}{9}\right) - 2\, sinc(3f)$$

**EX. 5** For the spectrum $G(f) = \frac{13}{4+j2\pi f}$. Find the Fourier transform of $x(t) = g(5t+3)$.

**solution** if $g(t) \xleftrightarrow{F.T.} G(f)$

$$g(5t) \xleftrightarrow{F.T.} \frac{1}{5} G\left(\frac{f}{5}\right)$$

and

$$g(t+t_0) \longleftrightarrow G(f)\, e^{j2\pi f t_0}$$

$$g(t+3) \longleftrightarrow G(f)\, e^{j2\pi f 3}$$

$$\therefore\ g(5t+3) \longleftrightarrow \frac{1}{5} G\left(\frac{f}{5}\right) e^{j2\pi f 3} = \frac{13/5}{4+j2\pi\frac{f}{5}}\, e^{j2\pi f 3}$$

**EX. 6** Find the Fourier transform of $\int_{-\infty}^{t} y(2t+4)\, dt$ when $Y(f) = \text{sinc}^3(f)$.

**solution** we know from the integration property that

$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \xleftrightarrow{F.T.} \frac{1}{j2\pi f} X(f)$$

$$\int_{-\infty}^{t} y(2t+4)\, dt \longleftrightarrow \frac{1}{2} \cdot \frac{1}{j2\pi f} \cdot Y\left(\frac{f}{2}\right) e^{j2\pi f 4}$$

$$\therefore\ Y(f) = \frac{e^{j2\pi f 4}}{j4\pi f}\, \text{sinc}^3\left(\frac{f}{2}\right)$$

EX. 7 what will be the Fourier transform of $\frac{d}{dt} g(2t)$

if you have $G(f) = rect^2(6f)$ ?

solution The differentiation Property shows that

$$\frac{d}{dt} g(t) \xleftrightarrow{F.T.} j2\pi f \; G(f)$$

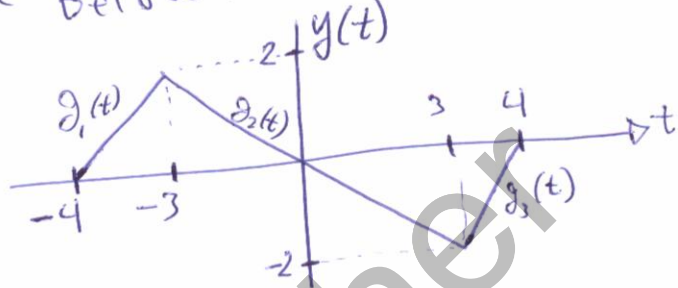
Since there is a time-scaling as well

$$g(\alpha t) \xleftrightarrow{F.T.} \frac{1}{|\alpha|} G\left(\frac{f}{\alpha}\right)$$

$$\therefore G(f) = \frac{1}{2} j2\pi f \; rect^2\left(\frac{6f}{2}\right)$$

$$= j\pi f \; rect^2(3f)$$

**EX.8** Find the Fourier transform of the signal shown in the Figure below.




**Solution**

we have three lines; $g_1$, $g_2$, and $g_3$

for $g_1 \Rightarrow$ the slope $a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{-3-(-4)}$

$$a = \frac{2}{-3+4} = \frac{2}{1} = 2$$

$\therefore g_1(t): \quad y - y_1 = a(x - x_1)$

$\quad\quad y - y_1 = a(t - t_1) \Rightarrow y - 0 = 2(t+4)$

$\therefore g_1(t) = y = 2t + 8$

For $g_2 \Rightarrow a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-2}{3+3} = -\frac{4}{6} = -\frac{2}{3}$

$y - y_1 = -\frac{2}{3}(t - t_1)$

$\therefore g_2(t): \quad y - 2 = -\frac{2}{3}(t+3) \rightarrow 3y \cancel{-6} = -2t \cancel{-6}$

$\therefore g_2(t) = y = \frac{1}{3}(-2t) = -\frac{2}{3}t$

For $g_3(t)$: $\quad a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0+2}{4-3} = 2$

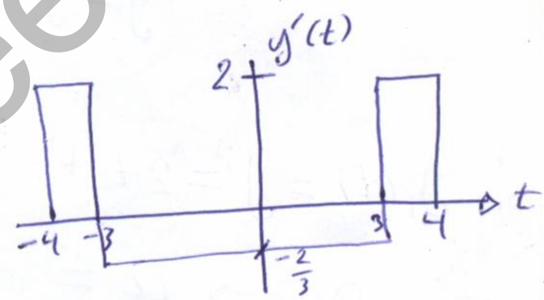
$\therefore g_3(t): \quad y - y_1 = 2(t - t_1) \Rightarrow y + 2 = 2(t-3)$

$\therefore g_3(t) = y = 2t - 8$

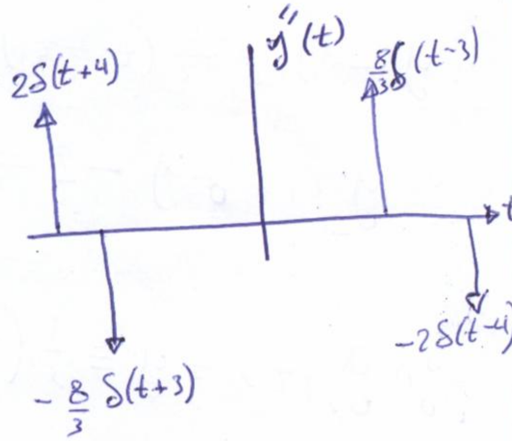
$$\therefore \quad y(t) = \begin{cases} g_1(t) & -4 \le t \le -3 \\ g_2(t) & -3 \le t \le 3 \\ g_3(t) & 3 \le t \le 4 \end{cases}$$

to get rid of the by-part integration, we can make use of the differentiation property to find the Fourier transform of $y(t)$.

# Taking $\frac{d}{dt} y(t) = \begin{cases} \frac{d}{dt} g_1(t) = 2 & -4 \le t \le -3 \\ \frac{d}{dt} g_2(t) = -\frac{2}{3} & -3 \le t \le 3 \\ \frac{d}{dt} g_3(t) = 2 & 3 \le t \le 4 \end{cases}$



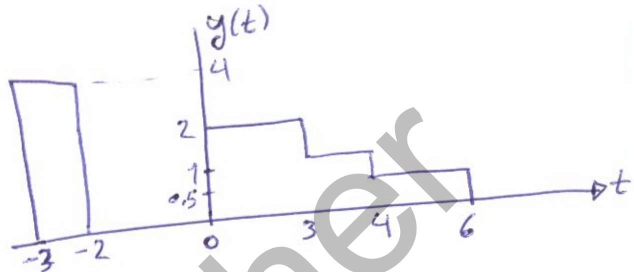

$$\frac{d^2}{dt^2} y(t) = \begin{cases} 2\delta(t+4) \\ \\ \end{cases}$$




$$y''(t) = 2\delta(t+4) - \frac{8}{3}\delta(t+3) + \frac{8}{3}\delta(t-3) - \frac{8}{3}\delta(t-4)$$

$$Y(f) = (j2\pi f)^2 \left[ 2e^{j2\pi f4} - \frac{8}{3} e^{j2\pi f3} + \frac{8}{3} e^{-j2\pi f3} - \frac{8}{3} e^{-j2\pi f4} \right]$$

## EX. 9 Find the Fourier transform of the signal shown below.




solution

$y(t) = g_1(t) + g_2(t) + g_3(t) + g_4(t)$

$g_1(t) = 4\,\mathrm{rect}\left(\frac{t+2.5}{1}\right)$

$g_2(t) = 2\,\mathrm{rect}\left(\frac{t-\frac{3}{2}}{3}\right)$

$g_3(t) = \mathrm{rect}\left(\frac{t-3.5}{1}\right)$

$g_4(t) = 0.5\,\mathrm{rect}\left(\frac{t-5}{2}\right)$

$$\therefore Y(f) = 4\,\mathrm{sinc}(f)\,e^{j2\pi f 2.5} + 6\,\mathrm{sinc}(3f)\,e^{-j2\pi f\frac{3}{2}} + \mathrm{sinc}(f)\,e^{-j2\pi f 3.5} + 0.5\,(2)\,\mathrm{sinc}(2f)\,e^{-j2\pi f 5}$$

$$= 4\,\mathrm{sinc}(f)\,e^{j2\pi f 2.5} + 6\,\mathrm{sinc}(3f)\,e^{-j2\pi f\frac{3}{2}} + \mathrm{sinc}(f)\,e^{-j2\pi f 3.5} + \mathrm{sinc}(2f)\,e^{-j2\pi f 5}$$

$$= 4\,\mathrm{sinc}(f)\,e^{j5\pi f} + 6\,\mathrm{sinc}(3f)\,e^{-j3\pi f} + \mathrm{sinc}(f)\,e^{-j7\pi f} + \mathrm{sinc}(2f)\,e^{-j10\pi f}$$

## EX. 10 Find the Fourier transform of $g(t) = 33\,e^{13t}\cos(\omega_0 t)\,u(t)$

Solution we have $\cos\omega_0 t \xleftrightarrow{F.T.} \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$

and we have $33\,e^{13t} \longleftrightarrow 33\,\frac{1}{13 - j2\pi f}$

$$\therefore G(f) = 33\left(\frac{1}{2}\right)\left[\frac{1}{13 - j2\pi(f-f_0)} + \frac{1}{13 - j2\pi(f+f_0)}\right]$$

**EX. 11** Find the Fourier transform of $g(t) = e^{-2t}u(t) + e^{2t}u(-t)$

**solution**

$$g(t) = e^{-2t}u(t) + e^{2t}u(-t)$$

$$= e^{-2t}u(t) + e^{2t}u(-t)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{1}{2+j2\pi f} \qquad \frac{1}{2-j2\pi f}$$

$$\therefore G(f) = \frac{4}{4+(2\pi f)^2}$$

**EX. 12** Use the duality theorem to find the Fourier transform of $\frac{1}{1+t^2}$

**solution** we know that $e^{-\alpha|t|} \xleftrightarrow{F.T.} \frac{2\alpha}{\alpha^2+\omega^2}$

the duality theorem states that

$$g(t) \longleftrightarrow G(f)$$

$$G(t) \longleftrightarrow g(-f)$$

**So that**

$$e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$$

$$\frac{2\alpha}{\alpha^2+t^2} \longleftrightarrow e^{-\alpha|-f|}$$

if $\alpha = 1$

$$2 \cdot \frac{1}{1+t^2} \longleftrightarrow \frac{1}{2}e^{-|f|}$$

EX.13 Find the Fourier transform of $g(t) = \cos[2\pi f_0 (t - t_0)]$

solution we know $g(t - t_0) \xleftrightarrow{F.T} G(f)\, e^{-j2\pi f t_0}$

$\therefore \cos[2\pi f_0 (t - t_0)] \xleftrightarrow{F.T.} \frac{1}{2} e^{-j2\pi f t_0} \left[ \delta(f - f_0) + \delta(f + f_0) \right]$.

EX.14 Find the inverse Fourier transform of

$$G(f) = \frac{2}{1 + 2j(2\pi(f - 10))}$$

solution

- It can be seen there is a frequency shift of 10 Hz. &
- and we know the standard form of $e^{-\alpha t} \longleftrightarrow \frac{1}{\alpha + j2\pi f}$
- we can write $G(f) = \frac{2}{2(\frac{1}{2} + j2\pi f)}$ / we have also the frequency shift

$\therefore g(t) = e^{-\frac{1}{2}t}\, e^{j2\pi 10 t}\, u(t)$

EX. 15 Use the Fourier transform pair

$$e^{-\alpha|t|} \xleftrightarrow{F.T} \frac{2\alpha}{\alpha^2+(2\pi f)^2}$$

Find the Fourier transform of $\dfrac{1}{4+t^2}$

Solution

$$e^{-\alpha|t|} \xleftrightarrow{\text{Fourier Transform}} \frac{2\alpha}{\alpha^2+(2\pi f)^2}$$

① ②

$$\frac{2\alpha}{\alpha^2+(2\pi t)^2} \longleftrightarrow e^{-\alpha|-f|}$$

if $\alpha=2$

$$\frac{1}{4+t^2} = \frac{\frac{4}{4}*1}{(2)^2+t^2} = \frac{1}{4}\,\frac{2*2}{2^2+t^2} \longrightarrow \frac{1}{4}e^{-2|f|}$$

in other words

$$\frac{1}{4+t^2} \xrightarrow{F.T.} \frac{1}{4}e^{-2|f|}$$

**EX. 16** Use Duality theorem to find the Fourier transform of $\frac{1}{4+t^2}\cos 2t$

**solution** $y(t) = f(t)\cos 2t$

$$F.T.\{f(t)\} = \frac{1}{4} e^{-2|f|}$$

Duality theorem

and we have

$$\cos\omega_o t = \frac{1}{2}[\delta(f-f_o)+\delta(f+f_o)]$$

$$\therefore\ Y(f) = \frac{1}{8} e^{-2|f-f_o|} + \frac{1}{8} e^{-2|f+f_o|}$$

but $f_o = \frac{1}{\pi}$ Hz

$$\therefore\ Y(f) = \frac{1}{8} e^{-2|f-\frac{1}{\pi}|} + \frac{1}{8} e^{-2|f+\frac{1}{\pi}|}$$

**EX. 1,7** Find the Fourier transform (using the duality theorem) of $6\ \mathrm{sinc}(3t)$.

**Solution** $g(t) = A\ \mathrm{rect}\left(\frac{t}{T}\right) \xrightarrow{F.T.} G(f) = AT\ \mathrm{sinc}(fT)$

we have $h(t) = 6\ \mathrm{sinc}(3t)$

$\therefore\ AT\ \mathrm{sinc}(fT)$

$AT = 6 \qquad T = 3$

$\therefore\ A = 2$

$\therefore\ H(f) = A\ \mathrm{rect}\left(\frac{-f}{T}\right) = 2\ \mathrm{rect}\left(\frac{f}{3}\right)$

**Pro. 1,8** Use the duality theorem to determine the Fourier transform of 5 sinc(4t).

**Solution**

$$A\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{1} AT\,\mathrm{sinc}(fT)$$

$$AT\,\mathrm{sinc}(tT) \xleftrightarrow{4} A\,\mathrm{rect}\left(\frac{-f}{T}\right)$$

(2) (3)

$\therefore\ 5\,\mathrm{sinc}(4t) = AT\,\mathrm{sinc}(fT)$

$\therefore\ \boxed{AT=5}\quad \boxed{T=4}$

$\therefore\ A = \frac{5}{4}$

$\therefore\ 5\,\mathrm{sinc}(4t) \longleftrightarrow \frac{5}{4}\,\mathrm{rect}\left(\frac{f}{4}\right)$

**Pro. 19** By the duality theorem, evalute the Fourier transform of 3 sinc(6t).

**Solution**

$$A\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\,\mathrm{Sinc}(fT)$$

$$AT\,\mathrm{sinc}(tT) \longleftrightarrow A\,\mathrm{rect}\left(\frac{-f}{T}\right)$$

But : $AT\,\mathrm{sinc}(tT) = 3\,\mathrm{sinc}(6t) \rightarrow \boxed{AT=3}\ \&\ \boxed{T=6}\ \therefore\ A=\frac{3}{6}=\frac{1}{2}$

$\therefore\ 3\,\mathrm{sinc}(6t) \xleftrightarrow{F.T.} \frac{1}{2}\,\mathrm{rect}\left(\frac{f}{6}\right)$

**Pro. 20** Find the value of $g(t) = \int_{-\infty}^{\infty} \text{sinc}^2(t)\, dt$.

**solution** By using the Parseval's theorem:

we know $\text{sinc}(t) \longleftrightarrow \text{rect}(f)$

and using parseval's theorem

$$\therefore \int_{-\infty}^{\infty} \text{sinc}^2(t)\, dt = \int_{-\infty}^{\infty} \text{rect}^2(f)\, df = \int_{-1/2}^{1/2} 1\, df = 1$$

**Pro. 21** Find the Fourier transform of $g(t) = e^{-\pi a^2 t^2}$.

**solution** $g(t) = e^{-\pi (at)^2}$, in other words, there is a time saling.

$$\therefore e^{-\pi (at)^2} \longleftrightarrow \frac{1}{|a|} e^{-\pi \left(\frac{f}{a}\right)^2}$$

Pro. 22 Plot the amplitude and phase spectrums of

$$G(f) = \frac{3}{4+jf}$$

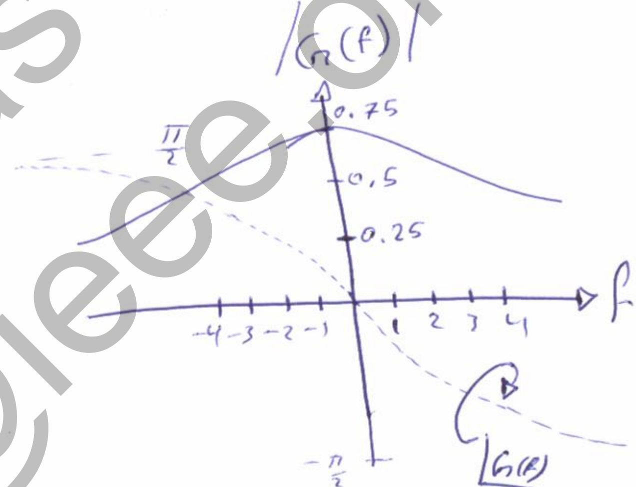
Solution

$$|G(f)| = \left|\frac{3}{4+jf}\right| = 3\left|\frac{1}{4+jf}\right| = \frac{3}{\sqrt{16+f^2}}$$

$$\angle G(f) = \tan^{-1}\frac{Img[G(f)]}{Real[G(f)]}$$

$$G(f) = \frac{3(4-jf)}{16+f^2} = \frac{12}{16+f^2} - j\frac{3f}{16+f^2}$$

$$\therefore \angle G(f) = \tan^{-1}\left(\frac{-3f}{12}\right) = \tan^{-1}\left(\frac{-f}{4}\right)$$



**Pro. 23** Given $g(t) = 3\cos(8\pi t) + 10\cos(25\pi t)$. Find the Fourier transform of $g(t - \frac{1}{30})$.

**solution**

$$g(t) = 3\cos(2\pi 4 t) + 10\cos(2\pi \tfrac{25}{2} t).$$

$$G(f) = 3(\tfrac{1}{2})\left[\delta(f-4) + \delta(f+4)\right] + \frac{10}{2}\left[\delta(f - \tfrac{25}{2}) + \delta(f + \tfrac{25}{2})\right].$$

we know $g(t - \frac{1}{30}) \xleftrightarrow{F.T.} G(f)\, e^{-j2\pi f \frac{1}{30}} = G(f)\, e^{-j\frac{\pi f}{15}}$

$$\therefore G(f) = \frac{3}{2}\left[\delta(f-4) + \delta(f+4)\right] e^{-j\frac{\pi f}{15}} + 5\left[\delta(f - \tfrac{25}{2}) + \delta(f + \tfrac{25}{2})\right] e^{-j\frac{\pi f}{15}}$$

**Pro. 24** calculate the convolution $g(t) = rect(t) * \cos(\pi t)$.

**solution** $rect(t) \xleftrightarrow{F.T.} sinc(f)$

$\cos(\pi t) \xleftrightarrow{F.T.} \frac{1}{2}\left[\delta(f - \frac{1}{2}) + \delta(f + \frac{1}{2})\right]$

$$G(f) = sinc(f)\, \tfrac{1}{2}\delta(f - \tfrac{1}{2}) + \tfrac{1}{2} sinc(f)\, \delta(f + \tfrac{1}{2})$$

$$= \tfrac{1}{2} sinc(\tfrac{1}{2})\, \delta(f - \tfrac{1}{2}) + \tfrac{1}{2} sinc(-\tfrac{1}{2})\, \delta(f + \tfrac{1}{2})$$

$sinc(\frac{1}{2}) = \frac{2}{\pi}$

$$= \frac{1}{2}\frac{2}{\pi}\delta(f - \tfrac{1}{2}) + \frac{1}{2}\frac{2}{\pi}\delta(f + \tfrac{1}{2}) = \frac{1}{\pi}\left[\delta(f - \tfrac{1}{2}) + \delta(f + \tfrac{1}{2})\right]$$

$$\therefore g(t) = \frac{2}{\pi}\cos(\pi t)$$

**Pro. 25** Determine the convolution $g(t) = 2\, rect(t) * \cos(2\pi t)$

**solution** $rect(t) \xleftrightarrow{F.T.} sinc(f)$ & $\cos(2\pi 1 t) \xleftrightarrow{F.T.} \frac{1}{2}\left[\delta(f-1) + \delta(f+1)\right]$

$$G(f) = 2\, sinc(f) \cdot \tfrac{1}{2}\left[\delta(f-1) + \delta(f+1)\right] = \underbrace{sinc(1)}_{0}\delta(f-1) + \underbrace{sinc(-1)}_{0}\delta(f+1)$$

$$\therefore g(t) = 0$$

**Pro. 26** What is the value of the energy of the signal $g(t) = \text{sinc}\left(\frac{t}{3}\right)$ ?

**Solution** using Parseval's theorem

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

$$= \int_{-\infty}^{\infty} |3\, \text{rect}(3f)|^2 df = 9 \int_{-\frac{1}{6}}^{\frac{1}{6}} \text{rect}(3f)\, df = 9 \int_{-\frac{1}{6}}^{1/6} df$$

$$E_g = 9\left[\frac{1}{6} + \frac{1}{6}\right] = 9\left[\frac{2}{6}\right] = \frac{9}{3} = 3\,J.$$

**Pro. 27** Evalute the energy of $2\,\text{sinc}^2(6t)$.

**Solution** By using Parseval's theorem;

and we know $\text{tri}(t) \xleftrightarrow{F.T.} \text{sinc}^2(f)$

$\text{tri}(\alpha t) \longleftrightarrow \frac{1}{|\alpha|}\text{sinc}^2\left(\frac{f}{\alpha}\right)$

$\text{sinc}^2(6f) \longrightarrow \text{tri}(\alpha f)$

$\text{sinc}^2(6f) - \text{sinc}^2\left(\frac{f}{\alpha}\right)$

$6 = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{6}$

$\text{sinc}^2(6f) \longleftrightarrow \frac{1}{6}\text{tri}\left(-\frac{f}{6}\right) = \frac{1}{6}\text{tri}\left(\frac{f}{6}\right)$

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df = \left(\frac{2}{6}\right)^2 \int_0^6 \text{tri}^2\left(\frac{f}{6}\right) df = \frac{4}{36}\int_0^6 \left(1 - \frac{f}{6}\right)^2 df$$

$$= \frac{1}{9}\int_0^6 \left[1 - \frac{f}{3} + \frac{f^2}{36}\right] df = \frac{1}{9}\left[f - \frac{f^2}{6} + \frac{f^3}{3*36}\right]_0^6 = \frac{1}{9}\left[6 - \frac{36}{6} + \frac{6*36}{3*36}\right]$$

$$= \frac{2}{9}\,J.$$

**Pro. 28** calculate the average value (total area under the curve) of $g(t)$, where

$$g(t) = 4\,\mathrm{sinc}\left(\frac{t-3}{6}\right).$$

**solution** $g(t)$ without time shift is $g(t) = 4\,\mathrm{sinc}\left(\frac{t}{6}\right)$

– using the duality theorem

$$A\,\mathrm{rect}\left(\frac{t}{T}\right) \longleftrightarrow AT\,\mathrm{sinc}(fT)$$

$$AT\,\mathrm{sinc}(tT) \longleftrightarrow A\,\mathrm{rect}\left(\frac{-f}{T}\right)$$

$$4\,\mathrm{sinc}\left(\frac{t}{6}\right) \qquad 24\,\mathrm{rect}(6f)$$

$\boxed{4 = AT}$ $\boxed{T = \frac{1}{6}}$

$$\therefore A = \frac{4}{T} = 24$$

$$\therefore G(f) = F.T.\left\{4\,\mathrm{sinc}\left(\frac{t-3}{6}\right)\right\} = 24\,\mathrm{rect}(6f)\,e^{-j6\pi f}$$

* Area under the curve is $G(0)$

$$\therefore \int_{-\infty}^{\infty} g(t)\,dt = G(0) = 24.$$

**Pro. 29** Find the Fourier transform of $x(t) = A[1 + a\,m(t)]\cos(\omega_0 t)$.

**solution** $x(t) = [A + Aa\,m(t)]\cos(\omega_0 t)$

$$x(t) = A\cos(\omega_0 t) + Aa\,m(t)\cos(\omega_0 t)$$

$$x(t) = x_1(t) + x_2(t)$$

$$x_1(t) = A\cos(\omega_0 t)$$

$$X_1(f) = \frac{A}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

$$x_2(t) = Aa\,m(t)\cos(\omega_0 t)$$

$$X_2(f) = \frac{Aa}{2}\left[M(f - f_0) + M(f + f_0)\right]$$

$$\therefore\ X(f) = X_1(f) + X_2(f)$$

$$X(f) = \frac{A}{2}\left[\delta(f + f_0) + \delta(f - f_0)\right] + \frac{Aa}{2}\left[M(f - f_0) + M(f + f_0)\right]$$

**Pro. 30** Find the area under $x(t) = 5\,\text{sinc}\left(\frac{t-4}{10}\right)$.

**Solution** area under $x(t) = K = \int_{-\infty}^{\infty} x(t)\,dt = \int_{-\infty}^{\infty} 5\,\text{sinc}\left(\frac{t-4}{10}\right)dt$

We know $5\,\text{sinc}\left(\frac{t}{10}\right) \longleftrightarrow AT\,\text{sinc}(fT)$

$\therefore\ AT = 5$ & $T = \frac{1}{10} \Rightarrow A = 50$

Hence $5\,\text{sinc}\left(\frac{t}{10}\right) \xleftrightarrow{F.T.} A\,\text{rect}\left(\frac{-f}{T}\right) = 50\,\text{rect}(10f)$

then $5\,\text{sinc}\left(\frac{t-4}{10}\right) \xleftrightarrow{F.T.} 50\,\text{rect}(10f)\,e^{-j8\pi f}$

$$K = \int_{-\infty}^{\infty} x(t)\,dt = X(0) = 50\,\text{rect}(0)\,e^{0} = 50.$$

Pro. 31 Find the area K under $g(t) = 12\,\mathrm{sinc}\left(\frac{t+3}{5}\right)$

Solution $K = \int_{-\infty}^{\infty} 12\,\mathrm{sinc}\left(\frac{t+3}{5}\right) dt$

Solving this integration is not easy. It can be evaluated but in other way. $G(0) = \int_{-\infty}^{\infty} 12\,\mathrm{sinc}\left(\frac{t+3}{5}\right) dt$

* Using Duality:-

$12\,\mathrm{sinc}\left(\frac{t}{5}\right) \xleftrightarrow{F.T.} AT\,\mathrm{sinc}(fT)$

$\therefore AT = 12 \quad \& \quad T = \frac{1}{5} \leftarrow A = 12 \times 5 = 60$

$A\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\,\mathrm{sinc}(fT)$

$AT\,\mathrm{sinc}(tT) \longleftrightarrow A\,\mathrm{rect}\left(\frac{-f}{T}\right)$

$12\,\mathrm{sinc}\left(\frac{t}{5}\right) \longleftrightarrow 60\,\mathrm{rect}(5f)$

also there is a time shift

$\therefore G(f) = 60\,\mathrm{rect}(5f)\,e^{j2\pi 6f}$

$K = G(0) = 60$

**Pro. 32** Find the area under $x(t) = 13\ \text{sinc}\left(\frac{t}{15}\right)$.

$\text{Area} = K = X(0)$

using Duality

$AT\ \text{sinc}(fT)$ —— $13\ \text{sinc}\left(\frac{t}{15}\right)$

$AT = 13$ & $T = \frac{1}{15} \Rightarrow A = 13 \times 15 = 195$

$A\ \text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\ \text{sinc}(fT)$

$AT\ \text{sinc}(tT) \xleftrightarrow{F.T.} A\ \text{rect}\left(\frac{-f}{T}\right)$

$13\ \text{sinc}\left(\frac{t}{15}\right) \longleftrightarrow 195\ \text{rect}(15f)$

$\therefore X(f) = 195\ \text{rect}(15f)$

Hence $X(0) = 195$.

Pro. 33 Find the area under K under $g(t) = x(4t-2)$, where $x(t) = 2\,\text{sinc}(t)$.

Solution. From Duality Property:

$$A\,\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\,\text{sinc}(fT)$$

$$AT\,\text{sinc}(tT) \xleftrightarrow{F.T.} A\,\text{rect}\left(\frac{-f}{T}\right)$$

$2\,\text{sinc}(t)$

$\therefore\ AT = 2 \rightarrow T = 1 \rightarrow A = 2$

Hence $X(f) = 2\,\text{rect}(f)$

$$x(4t-2) \longleftrightarrow \frac{1}{4} X\left(\frac{f}{4}\right) e^{-j2\pi f 2}$$

$$\therefore\ G(f) = \frac{1}{4} e^{-j2\pi 2f}\left[2\,\text{rect}\left(\frac{f}{4}\right)\right]$$

$$G(f) = \frac{1}{2} e^{-j4\pi f}\,\text{rect}\left(\frac{f}{4}\right)$$

$$K = G(0) = \frac{1}{2}$$

**Pro. 34** Find the Fourier transform of: ① $x(t)$, ② $x(2(t-1))$, and ③ $x(2t-1)$, when $x(t) = 10\sin(2\pi 2t)$.

**solution** $\sin(2\pi f_o t) \longleftrightarrow \frac{1}{j2}\left[\delta(f-f_o) - \delta(f+f_o)\right]$

$\therefore \; 10\sin(2\pi 2t) \longleftrightarrow \frac{10}{j2}\left[\delta(f-2) - \delta(f+2)\right]$

① $\therefore X(f) = -j5\left[\delta(f-2) - \delta(f+2)\right]$.

② $x(2(t-1)) \xleftrightarrow{F.T.} \frac{1}{2} X\left(\frac{f}{2}\right) e^{-j2\pi f}$

$\therefore \; X(f) = \frac{-j5}{2}\left[\delta\left(\frac{f}{2}-2\right) - \delta\left(\frac{f}{2}+2\right)\right] e^{-j2\pi f}$

we know $\delta(\alpha f) = \frac{1}{\alpha}\delta(f) \rightarrow \boxed{\alpha = \frac{1}{2}}$

$\therefore \; X(f) = -j5\left[\delta(f-4) - \delta(f+4)\right] e^{-j2\pi f}$

③ $x(2t-1) \xleftrightarrow{F.T.} \frac{1}{2} X\left(\frac{f}{2}\right) e^{-j2\pi \frac{f}{2}}$ [each $f$ replaced by $\frac{f}{2}$]

$X(f) = \frac{10}{j4} e^{-j2\pi\frac{f}{2}}\left[\delta\left(\frac{f}{2}-2\right) - \delta\left(\frac{f}{2}+2\right)\right]$

$X(f) = \frac{20}{j4} e^{-j2\pi\frac{f}{2}}\left[\delta(f-4) - \delta(f+4)\right]$

$X(f) = -j5\, e^{-j2\pi\frac{f}{2}}\left[\delta(f-4) - \delta(f+4)\right]$

**Pro. 35** What is the Fourier transform of the convolution of $x(t) = 3\sin(2\pi t)$ with $g(t) = 5\,\delta(t-3)$?

**solution**

$$x(t) * g(t) = 3\sin(2\pi t) * 5\,\delta(t-3)$$

$$z(t) = 15\sin(2\pi(t-3)).$$

$$Z(f) = 15\,\frac{1}{j2}\left[\delta(f-1) - \delta(f+1)\right] e^{-j6\pi f}$$

$$Z(f) = \frac{15}{j2}\left[\delta(f-1) - \delta(f+1)\right] e^{-j6\pi f}$$

OR [second solution]

$$X(f) = \frac{3}{j2}\left[\delta(f-1) - \delta(f+1)\right]$$

$$G(f) = 5\,e^{-j6\pi f}$$

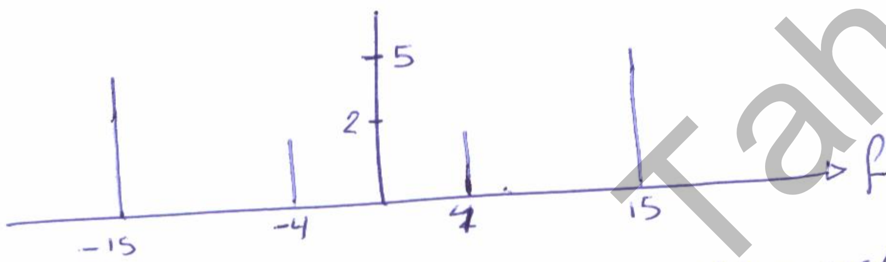
$$Z(f) = X(f)\,G(f)$$

$$Z(f) = \frac{15}{j2}\left[\delta(f-1) - \delta(f+1)\right] e^{-j6\pi f}$$

**Pro. 36** Plot the amplitude and phase spectrums of the signal $g(t) = 4\cos(8\pi t) + 10\cos(30\pi t)$.

**solution**

$$G(f) = 2\left[\delta(f-4) + \delta(f+4)\right] + 5\left[\delta(f-15) + \delta(f+15)\right]$$




**Pro. 37** Find the convolution of $g(t) = rect(t) * \cos(\pi t)$.

$\omega_0 = 2\pi f_0 = \pi \rightarrow f_0 = \frac{1}{2}$

**solution** $g(t) = g_1(t) * g_2(t)$

$G_1(f) = sinc(f)$ & $G_2(f) = \frac{1}{2}\left[\delta(f-\frac{1}{2}) + \delta(f+\frac{1}{2})\right]$

$$G(f) = sinc(f)\,\frac{1}{2}\left[\delta(f-\tfrac{1}{2}) + \delta(f+\tfrac{1}{2})\right]$$

$$= \frac{1}{2}\left[sinc(\tfrac{1}{2})\,\delta(f-\tfrac{1}{2}) + sinc(-\tfrac{1}{2})\,\delta(f+\tfrac{1}{2})\right]$$

$$= \frac{1}{2}\left\{\frac{\sin(\pi\frac{1}{2})}{\pi\frac{1}{2}}\,\delta(f-\tfrac{1}{2}) + \frac{\sin(\frac{-\pi}{2})}{-\pi/2}\,\delta(f+\tfrac{1}{2})\right\}$$

$$= \frac{1}{2}\left[\frac{2}{\pi}\,\delta(f-\tfrac{1}{2}) + \frac{2}{\pi}\,\delta(f+\tfrac{1}{2})\right]$$

$$= \frac{2}{\pi}\left[\delta(f-\tfrac{1}{2}) + \delta(f+\tfrac{1}{2})\right]$$

$$\therefore \boxed{g(t) = \frac{2}{\pi}\cos(\pi t)}$$

**P. 37** calculate the convolution of $g(t) = rect(t)$ with $h(t) = \cos(2\pi t)$.

solution $f(t) = g(t) * h(t) = rect(t) * \cos(2\pi t)$.

$G(f) = sinc(f)$ & $H(f) = \frac{1}{2}[\delta(f-1) + \delta(f+1)]$.

$$F(f) = sinc(f)\,\frac{1}{2}[\delta(f-1) + \delta(f+1)]$$

$$= \frac{1}{2}[\underbrace{\cancel{sinc(+1)}}_{zero}\,\delta(f-1) + \underbrace{\cancel{sinc(-1)}}_{zero}\,\delta(f+1)]$$

$F(f) = 0$

$\therefore\ f(t) = 0$

**P. 38** Find the convolution of $y(t) = sinc(\frac{t}{2})$ with $x(t) = sinc(t)$.

solution $A\,rect(\frac{t}{T}) \longleftrightarrow AT\,sinc(fT) \Rightarrow sinc(\frac{t}{2})$ $AT = 1$ & $T = \frac{1}{2}$ $\boxed{A = 2}$

$\therefore\ sinc(\frac{t}{2}) \longleftrightarrow 2\,rect(2f)$

and $sinc(t) \longleftrightarrow rect(f)$

$\therefore\ Y(f) = 2\,rect(2f)$ & $X(f) = rect(f)$

$g(t) = y(t) * x(t) \overset{F.T.}{\Longleftrightarrow} Y(f)\cdot X(f) = 2\,rect(2f)\,rect(f) = 2\,rect(2f)$

$\therefore\ g(t) = I.F.T.\{2\,rect(2f)\}$

$A\,rect(\frac{t}{T}) \xleftrightarrow{F.T.} AT\,sinc(fT)$

$AT\,sinc(tT) \longleftrightarrow A\,rect(\frac{-f}{T})$

$\underbrace{2\,rect(2f)}$ $\Rightarrow A = 2$ & $T = \frac{1}{2}$

$\therefore\ g(t) = \frac{2}{2}\,sinc(\frac{t}{2}) = sinc(\frac{t}{2})$

**P.39** Convolve $g_1(t) = \text{sinc}(t)$ with $g_2(t) = \text{sinc}^2(\frac{t}{2})$

**solution** $g(t) = g_1(t) * g_2(t) = \text{sinc}(t) * \text{sinc}^2(\frac{t}{2})$

$G_1(f) = \text{rect}(f)$ & $G_2(f) = \text{tri}(2f)$

$A\,\text{tri}(\frac{t}{T}) \xleftrightarrow{F.T.} AT\,\text{sinc}^2(fT)$

$\text{sinc}^2(\frac{t}{2}) \Leftarrow T = 1/2$ & $A = 1$

$AT\,\text{sinc}^2(tT) \longrightarrow A\,\text{tri}(\frac{-f}{T})$

$\therefore F.T.\{\text{sinc}^2(\frac{t}{2})\} = \text{tri}(2f)$

$\therefore G(f) = G_1(f)\,G_2(f) = \text{rect}(f)\,\text{tri}(2f) = \text{tri}(2f)$

$\therefore g(t) = \text{sinc}^2(\frac{t}{2})$

**P.40** Evalute the convolution of $e^{-t}u(t)$ with $\sin(2\pi t)$.

**solution** $g(t) = g_1(t) * g_2(t) = e^{-t}u(t) * \sin(2\pi t)$

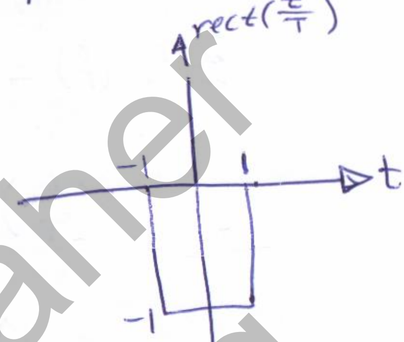
$G_1(f) = \frac{1}{1+j2\pi f}$ & $G_2(f) = \frac{1}{j2}[\delta(f-1) - \delta(f+1)]$

$G(f) = G_1(f) \cdot G_2(f) = \frac{1}{1+j2\pi f}[\frac{1}{j2}\delta(f-1) - \frac{1}{j2}\delta(f+1)]$

$= \frac{1}{j2}\left[\frac{\delta(f-1)}{1+j2\pi} - \frac{\delta(f+1)}{1-j2\pi}\right] =$

$$= \frac{\cos(2\pi t - 2.984)}{\sqrt{1+(2\pi)^2}}$$

P.41 Find the Fourier transform of $g(t) = \delta(t-1) - \delta(t+1)$, by using the differentiation property and the rectangular function.




Solution

$$\underbrace{\frac{d}{dt}\,\mathrm{rect}\left(\frac{t}{T}\right)} = \delta(t-1) - \delta(t+1) = g(t)$$

$T = 2$

$A = 1$

$$\Downarrow$$

$$G(f) = -j2\pi f\,T\,\mathrm{sinc}(fT) = -j2\pi f\,2\,\mathrm{sinc}(2f)$$

$$= -\cancel{j}\cancel{2\pi f}\,2\,\frac{\sin(2\pi f)}{2\pi f} = -j2\sin(2\pi f)$$

also $g(t) = \delta(t-1) - \delta(t+1) \overset{F.T.}{\longleftrightarrow} e^{-j2\pi f} - e^{j2\pi f} = G(f)$

$$\therefore\ G(f) = \frac{j2}{j2}\left[-e^{j2\pi f} + e^{-j2\pi f}\right]$$

$$= -\frac{j2}{j2}\left[e^{j2\pi f} - e^{-j2\pi f}\right]$$

$$= -j2\sin(2\pi f)$$

P. 42 Find the inverse Fourier transform of $2\cos(2\pi f)$.

Solution $G(f) = 2\cos(2\pi f) = \frac{2}{2}\left[e^{j2\pi f} + e^{-j2\pi f}\right] = e^{j2\pi f} + e^{-j2\pi f}$

$g(t) = \delta(t+1) + \delta(t-1)$

P. 43 Find the inverse Fourier transform of $4\cos(8\pi f)$

Solution $G(f) = 4\cos(8\pi f) = 4\,\frac{1}{2}\left[e^{j2\pi 4f} + e^{-j2\pi 4f}\right]$

$\therefore\ g(t) = 2\delta(t+4) + 2\delta(t-4)$.

P. 44 Find the inverse Fourier transform of $1-\cos^2(\pi f)$.

Solution $G(f) = 1-\cos^2(\pi f) = \sin^2(\pi f)$

$G(f) = \frac{1}{2}\{1-\cos(2\pi f)\}$

$G(f) = \frac{1}{2} - \frac{1}{2}\cos(2\pi f) = \frac{1}{2} - \frac{1}{2}\left[\frac{1}{2}e^{j2\pi f} + \frac{1}{2}e^{-j2\pi f}\right]$

$g(t) = \frac{1}{2}\delta(t) - \frac{1}{4}\left[\delta(t+1) + \delta(t-1)\right]$

P. 45 Find the inverse Fourier transform of $8\sin(10\pi f)$.

Solution $G(f) = 8\sin(2\pi 5f) = \frac{8}{j2}\left[e^{j2\pi f5} - e^{-j2\pi f5}\right]$

$g(t) = -j4\left[\delta(t+5) - \delta(t+5)\right]$

**P. 46** Find the inverse fourier transform of $2\,\mathrm{tri}\left(\frac{f}{2}\right)e^{-j6\pi f}$

**Solution** $G(f) = 2\,\mathrm{tri}\left(\frac{f}{2}\right)e^{-j6\pi f}$

we know that $A\,\mathrm{tri}\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\,\mathrm{sinc}^2(fT)$

$AT\,\mathrm{sinc}^2(tT) \xleftrightarrow{F.T.} A\,\mathrm{tri}\left(\frac{-f}{T}\right)$

$2\,\mathrm{tri}\left(\frac{f}{2}\right) \Rightarrow A = 2 \;\&\; T = 2$

$\therefore\; g_1(t) = 4\,\mathrm{sinc}^2(2t)$

but there is a phase shift $\left[e^{-j6\pi f}\right]$

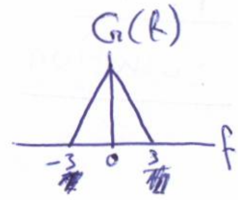
$e^{-j6\pi f} = e^{-j2\pi f 3}$

$\therefore\; g(t) = 4\,\mathrm{sinc}^2(2(t-3))$

**P. 47** calculate the energy of $2\,\mathrm{sinc}^2(3t)$.

**Solution** Using Parseval's theorem

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2\,dt = \int_{-\infty}^{\infty} |G(f)|^2\,df$$




$2\,\mathrm{sinc}^2(3t) \xrightarrow{F.T.} \frac{2}{3}\,\mathrm{tri}\left(\frac{f}{3}\right)$ $\quad\therefore\; E_g = \int_{-3}^{3} \left|\frac{2}{3}\,\mathrm{tri}\left(\frac{f}{3}\right)\right|^2 df$

* since $G(f)$ is even, and shifting it to the right to start at $f=0$, and since the energy of the first half = energy of second half due to symmetry, hence

$$E_g = \left(\frac{2}{3}\right)^2 2\int_0^3 \left(\frac{f}{3}\right)^2 df = \frac{8}{9}.$$

**P.48** Find the inverse Fourier transform of $X(f) = -22\,\mathrm{rect}\left(\frac{f}{5}\right)$.

**solution** $X(f) = -22\,\mathrm{rect}\left(\frac{f}{5}\right)$

$$A\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\,\mathrm{sinc}(fT)$$

$$AT\,\mathrm{sinc}(tT) \xleftrightarrow{F.T.} A\,\mathrm{rect}\left(\frac{f}{T}\right)$$

$-22\,\mathrm{rect}\left(\frac{f}{5}\right)$ ∴ $A = -22$, $T = 5$

$$\therefore\ x(t) = -22 \times 5\ \mathrm{sinc}(5t) = -110\,\mathrm{sinc}(5t)$$

**P.49** what is the inverse Fourier transform of $X(f) = \frac{\mathrm{sinc}(-20f)}{30}$.

**solution** $X(f) = \frac{1}{30}\,\mathrm{sinc}(-20f) = \frac{1}{30}\,\mathrm{sinc}(20f)$

$AT = \frac{1}{30}$, $T = 20$

$A = \frac{1}{30} \cdot \frac{1}{20} = \frac{1}{600}$

$$x(t) = \frac{1}{600}\,\mathrm{rect}\left(\frac{t}{20}\right)$$

**P.50** Determine the FT of $g(t) = \delta(t-2) * \cos(2\pi 10t)$.

**solution** $g(t) = \delta(t-2) * \cos(2\pi 10 t) = g_1(t) * g_2(t)$

$$G_1(f) = e^{-j4\pi f} \quad \& \quad G_2(f) = \frac{1}{2}\left[\delta(f-10) + \delta(f+10)\right]$$

$$G(f) = G_1(f)\,G_2(f) = \frac{1}{2}\,e^{-j4\pi f}\left[\delta(f-10) + \delta(f+10)\right].$$

**P. 51** Find the FT of $g(t) = \dfrac{\text{sinc}^2\left(\frac{t-1}{4}\right)}{15}$ and plot the amplitude and phase spectrums.

**solution** $g(t) = \frac{1}{15}\,\text{sinc}^2\left(\frac{t-1}{4}\right)$

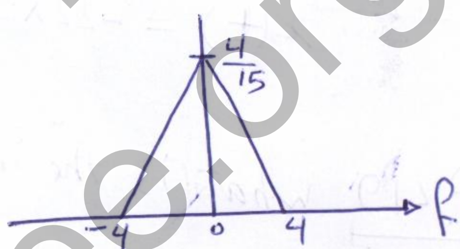
$A\,\text{tri}\left(\frac{t}{T}\right) \xleftrightarrow{F.T.} AT\,\text{sinc}^2(fT)$ $\qquad \frac{1}{15}\,\text{sinc}^2\left(\frac{t}{4}\right)$

$AT\,\text{sinc}^2(tT) \xleftrightarrow{F.T.} A\,\text{tri}\left(\frac{f}{T}\right)$

$\frac{1}{15}\,\text{sinc}^2\left(\frac{t}{4}\right) \Rightarrow \therefore\ AT = \frac{1}{15}\ \ \&\ T = \frac{1}{4} \Rightarrow A = \frac{1}{15}\,\frac{1}{T} = \frac{4}{15}$

$\therefore\ \frac{1}{15}\,\text{sinc}^2\left(\frac{t}{4}\right) \xleftrightarrow{F.T.} \frac{4}{15}\,\text{tri}(4f)$

Thus $G(f) = \frac{4}{15}\,\text{tri}(4f)\,e^{-j2\pi f}$




$\angle G(f) = \tan^{-1}\left(\frac{-\sin 2\pi f}{\cos 2\pi f}\right) = -\tan^{-1}(\tan(2\pi f)) = -2\pi\omega$