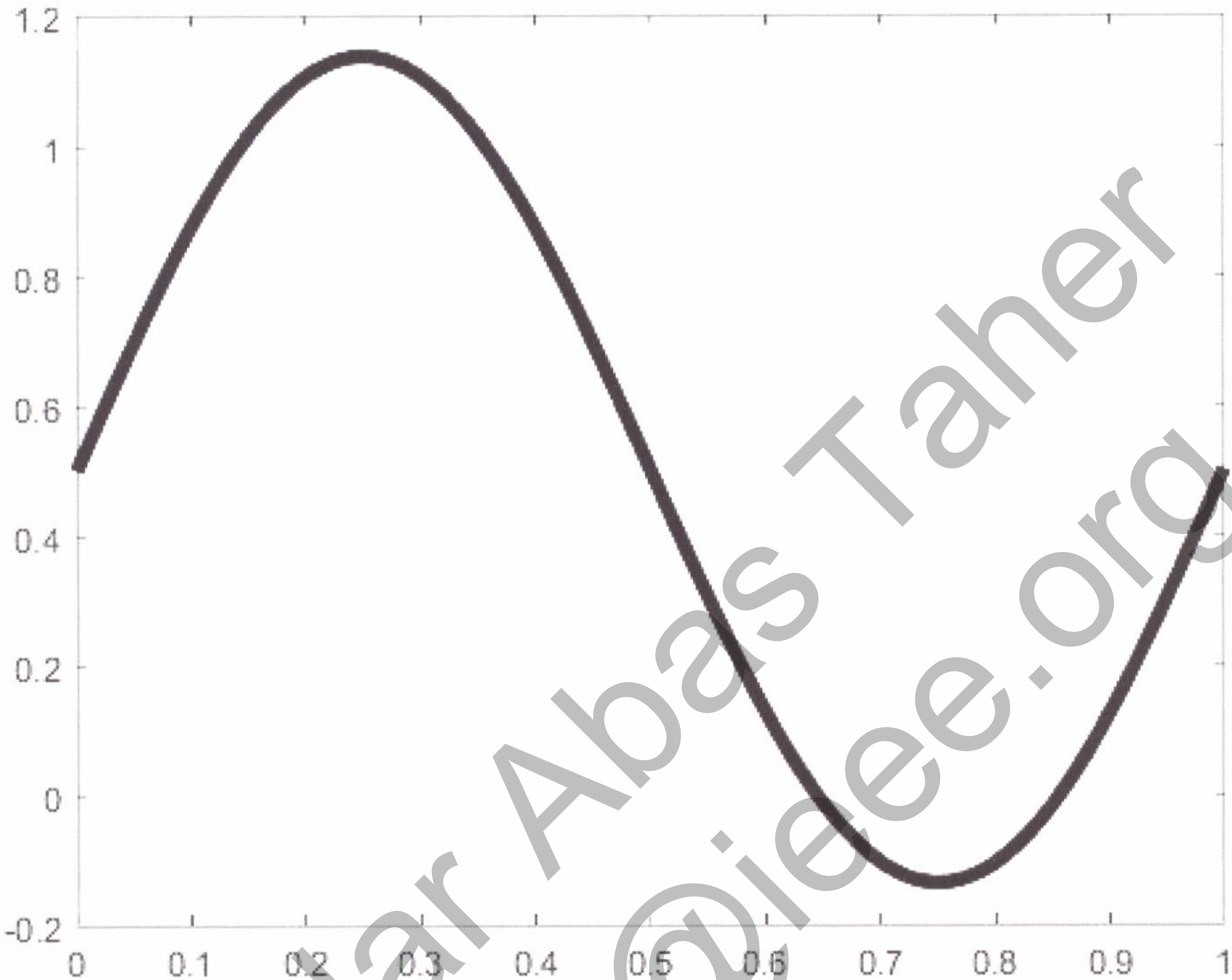
Fundamentals of Communications Engineering

Department of Communications Engineering, College of Engineering, University of Diyala, 2016-2017

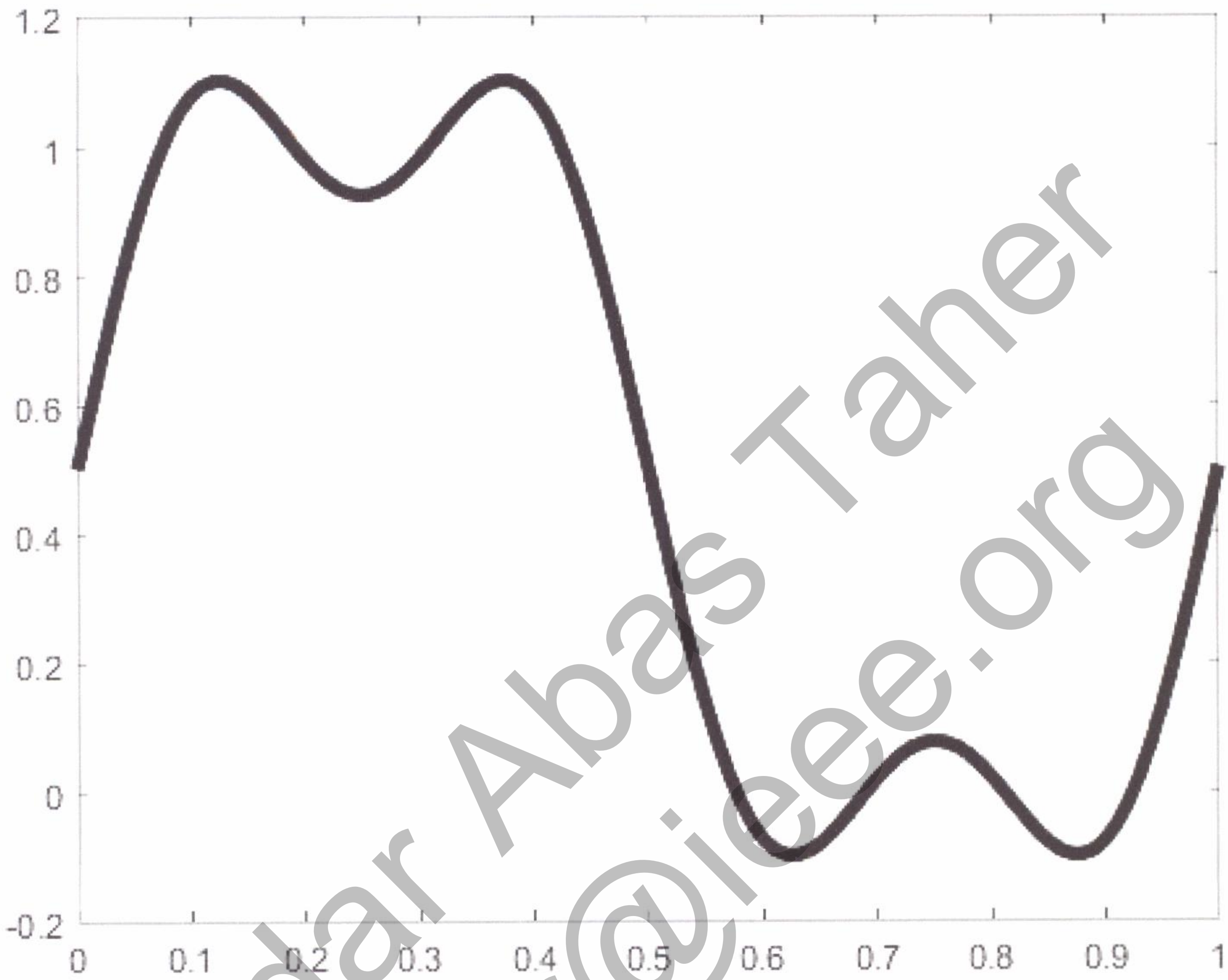
Class: Second Year
Instructor: Dr. Montadar Abas Taher
Room: Comm-02

Lecture: 05

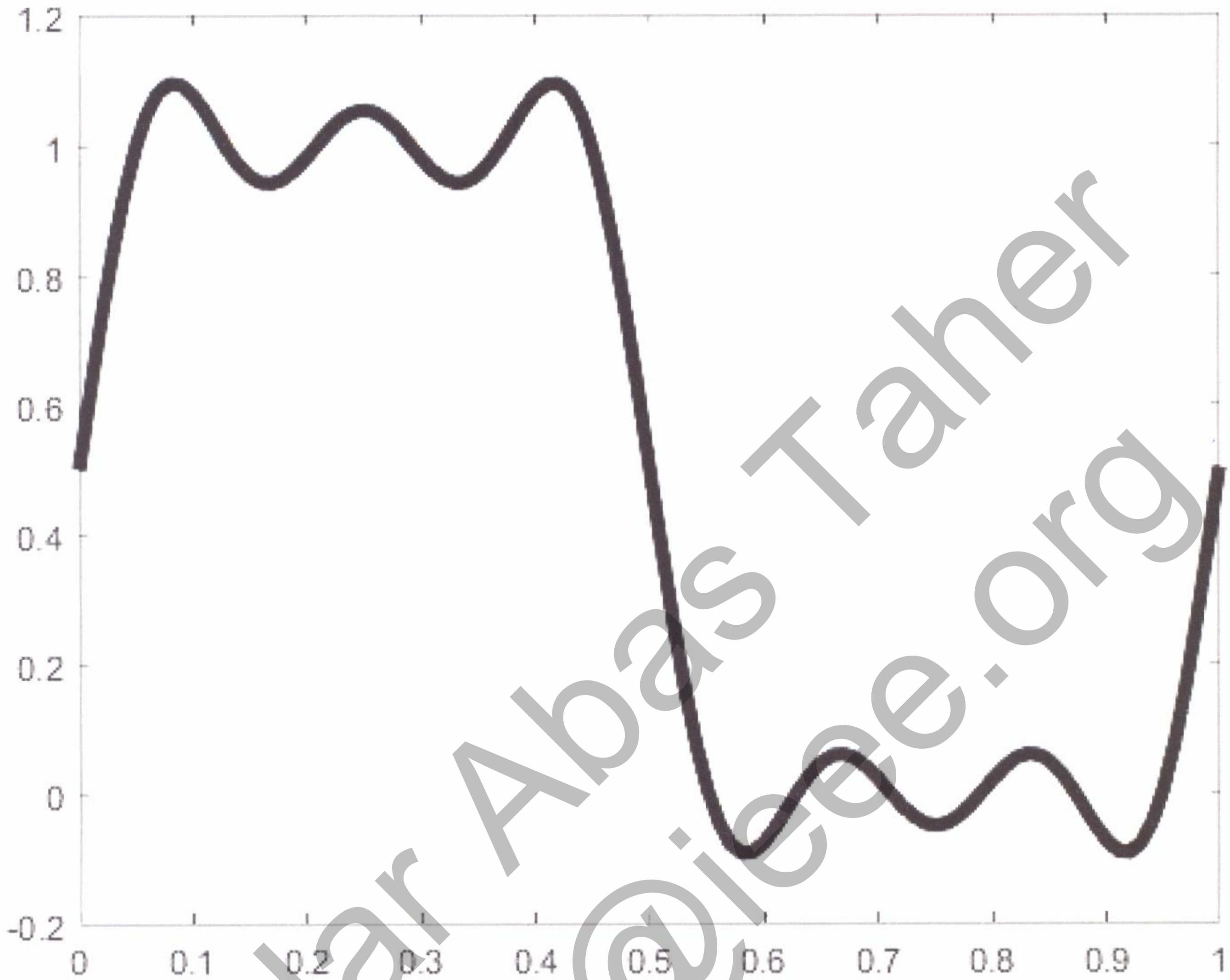
Fourier Series:



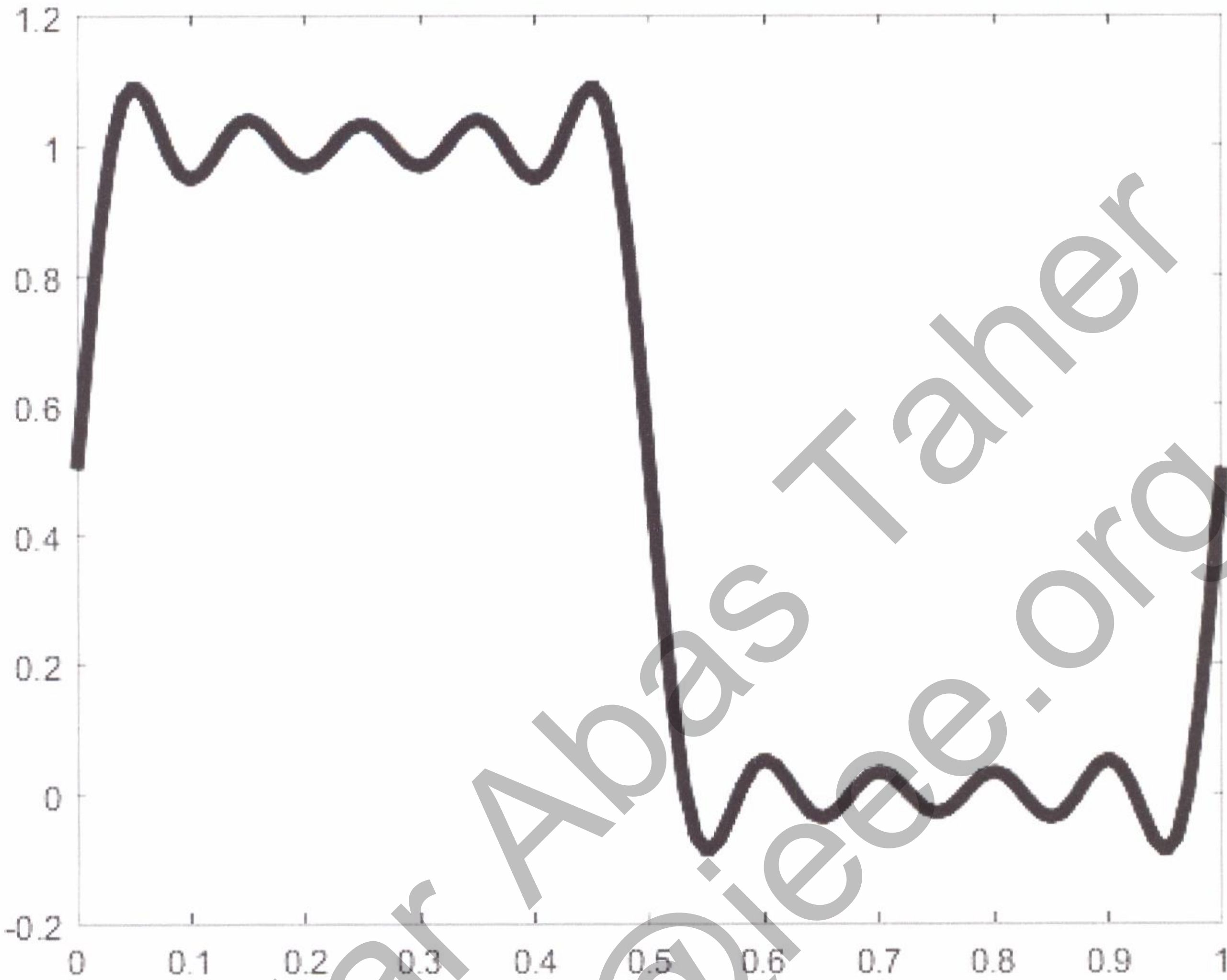
$$f(t) = a\sin(2\pi 1t)$$



$$f(t) = a_1 \sin(2\pi 1 t) + a_2 \sin(2\pi 2 t) + a_3 \sin(2\pi 3 t)$$



$$f(t) = a_1 \sin(2\pi 1t) + a_2 \sin(2\pi 2t) + a_3 \sin(2\pi 3t) + a_4 \sin(2\pi 4t) + a_5 \sin(2\pi 5t)$$



$$f(t) = a_1 \sin(2\pi 1t) + a_2 \sin(2\pi 2t) + a_3 \sin(2\pi 3t) + \cdots + a_{10} \sin(2\pi 10t)$$

$$f(t) = a_1 \sin(2\pi 1t) + a_2 \sin(2\pi 2t) + a_3 \sin(2\pi 3t) + \cdots + a_{1000} \sin(2\pi 1000t)$$

Fourier Series

The periodic function which satisfys Dirichlet conditions can be expressed as,

$$f(t) = a_0 + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + \dots + a_n \cos(n\omega t)$$
$$+ b_1 \sin(\omega t) + b_2 \sin(2\omega t) + \dots + a_n \sin(n\omega t)$$

OR

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(\omega n t) + b_n \sin(\omega n t) \right]$$

where $a_0 = \frac{1}{T} \int_0^T f(t)\, dt$ = Average value, OR DC value

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t)\, dt$$

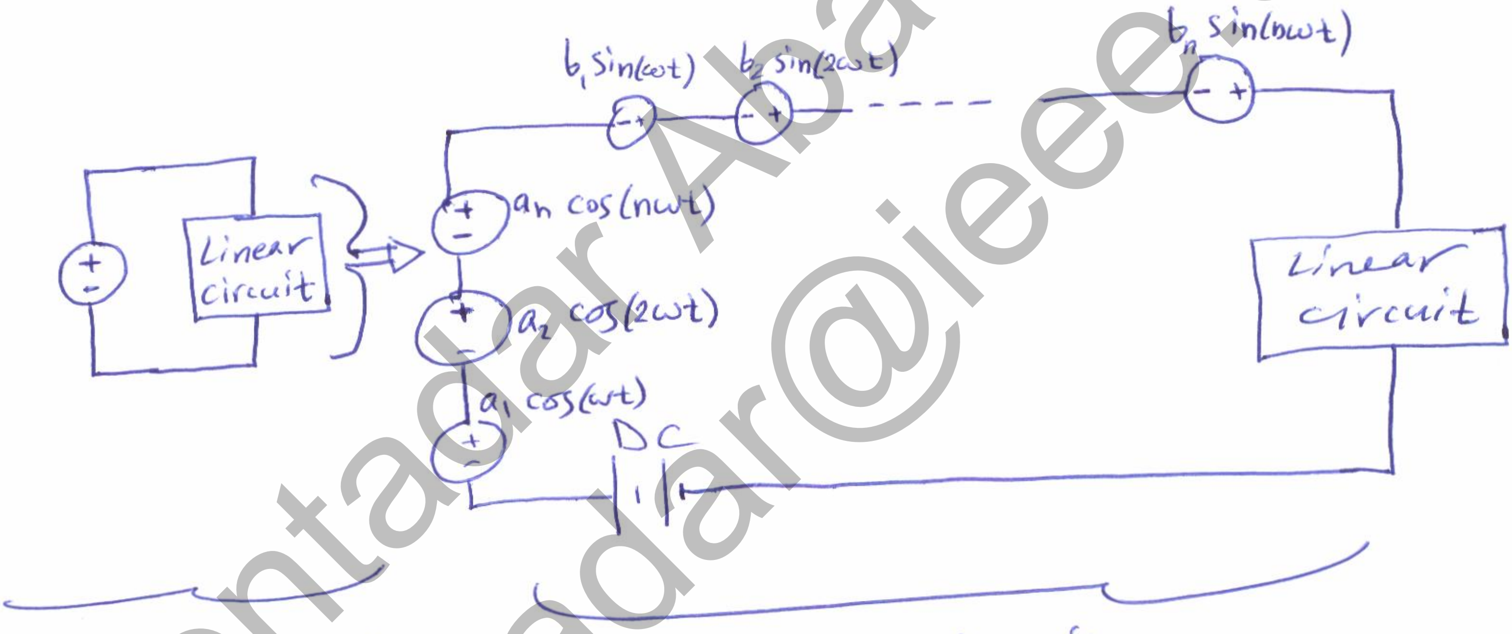
* $\left.\begin{matrix} a_0 \\ a_n \\ b_n \end{matrix}\right\}$ are called Fourier series coefficients.

Dirichlet conditions :- are

1) The signal has a finite number of discontinuities.

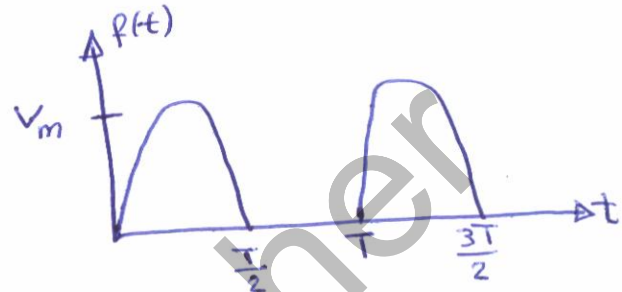
2) The signal has a finite number of maxima and minima.

3) The integral $\int_0^T |f(t)|\,dt$ is finite.



original circuit   Equivalent circuit

EX. Find the triganometric Fourier series for the half-wave rectified sine wave shown below.



Solution

$$f(t) = \begin{cases} V_m \sin(\omega t) & 0 \leq t \leq \frac{T}{2} \\ 0 & \frac{T}{2} \leq t \leq T \end{cases}$$

$$a_0 = D.C. = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\left[\int_0^{T/2} f(t)\,dt + \int_{T/2}^{T} f(t)\,dt\right]$$

$$= \frac{1}{T}\int_0^{T/2} V_m \sin(\omega t)\,dt = \frac{V_m}{T}\int_0^{T/2} \sin(\omega t)\,dt = \frac{-V_m}{\omega T}\left[\cos(\omega t)\right]_0^{T/2}$$

$$\therefore \boxed{a_0 = \frac{V_m}{\pi}}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt = \frac{2V_m}{T}\int_0^{T/2} \sin(\omega t)\cos(n\omega t)\,dt$$

$$= \frac{2V_m}{T}\int_0^{T/2} \left[\sin((n+1)\omega t) - \sin((n-1)\omega t)\right]dt$$

$$\therefore a_n = \frac{2V_m}{\omega T}\left[-\frac{\cos\left[(n+1)\frac{\omega T}{2}\right]}{n+1} + \frac{1}{n+1} + \frac{\cos\left[(n-1)\frac{\omega T}{2}\right]}{n-1} - \frac{1}{n-1}\right]$$

$$= \frac{V_m}{\pi}\left[\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{\cos[(n-1)\pi]}{n-1} - \frac{\cos[(n+1)\pi]}{n+1}\right)\right]$$

$$= \frac{V_m}{\pi}\left[\frac{2}{1-n^2} + \frac{2\sin(n\pi)\sin(\pi) + 2\cos(n\pi)\cos(\pi)}{n^2-1}\right] = \frac{2V_m}{\pi}\left[\frac{1}{1-n^2} + \frac{\cos(n\pi)}{n^2-1}\right]$$

$$\therefore \boxed{a_n = \frac{2V_m(1+\cos(n\pi))}{\pi(1-n^2)} \qquad n \neq \mp 1}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt = \frac{2V_m}{T}\int_0^{T/2}\sin(\omega t)\sin(n\omega t)\,dt$$

$$= \frac{2V_m}{T}\int_0^{T/2}\left[\cos[(n-1)\omega t] - \cos[(n+1)\omega t]\right]dt$$

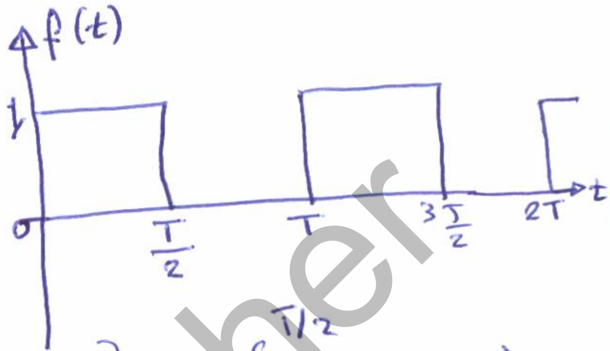
$$= \frac{2V_m}{\omega T}\left[\frac{\sin[(n-1)\omega t]}{n-1} - \frac{\sin[(n+1)\omega t]}{n+1}\right]_0^{T/2}$$

$$= \frac{V_m}{\pi}\left[\frac{\sin((n-1)\pi)}{n-1} - \frac{\sin((n+1)\pi)}{n+1}\right] = 0 \quad n \neq 1$$

$$\therefore \boxed{b_n = 0 \qquad n \neq 1}$$

$$\therefore \boxed{f(t) = \frac{V_m}{\pi} + \sum_{n=2}^{\infty}\frac{2V_m(1+\cos(n\pi))}{\pi(1-n^2)}}$$

EX. Find the trigonometric Fourier series for the wave shown below.



$$f(t)=\begin{cases}1 & \text{for } 0\leq t\leq \frac{T}{2}\\ 0 & \text{for } \frac{T}{2}\leq t<T\end{cases}$$

$$a_0=\frac{1}{T}\int_0^T f(t)dt=\frac{1}{T}\left[\int_0^{T/2} f(t)\,dt+\int_{T/2}^{T} f(t)\,dt\right]=\frac{1}{T}\int_0^{T/2} f(t)\,dt=\frac{1}{2}$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt=\frac{2}{T}\int_0^{T/2}\cos(n\omega t)\,dt=\frac{2}{n\omega T}\Big[\sin(n\omega t)\Big]_0^{T/2}$$

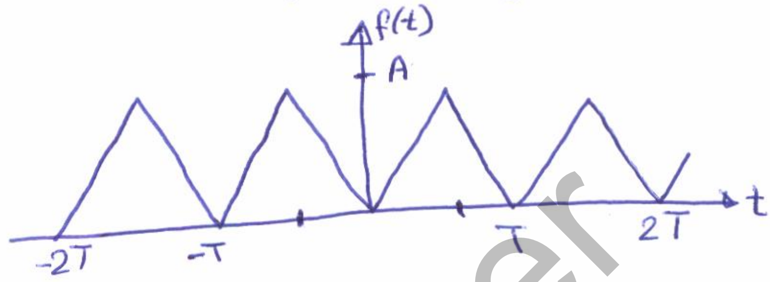
$$a_n=\frac{1}{n\pi}\left[\sin(n\pi)-\sin(0)\right]=0$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt=\frac{2}{T}\int_0^{T/2}\sin(n\omega t)\,dt=\frac{2}{n\omega T}\Big[\cos(n\omega t)\Big]_0^{T/2}$$

$$=\frac{2}{n\pi}\left[\cos(n\pi)-\cos(0)\right]=-\frac{1}{n\pi}\left[(-1)^n-1\right]$$

$$\therefore b_n=\begin{cases}\frac{2}{n\pi} & n \text{ is odd}\\ 0 & n \text{ is even}\end{cases}$$

EX. Find the Fourier series coefficients for a triangular wave shown below.



solution

$$f(t) = \begin{cases} -\frac{2A}{T}t & -\frac{T}{2} \leq t < 0 \\ \frac{2A}{T} & 0 \leq t < \frac{T}{2} \end{cases}$$

NOTE: if $f(t)$ even then $b_n = 0$, and if $f(t)$ is odd then $a_n = 0$.

$$a_0 = \frac{1}{T}\int_{-T/2}^{0} -\frac{2At}{T}dt + \frac{1}{T}\int_{0}^{T/2}\frac{2A}{T}t\,dt$$

$$= \frac{2A}{T^2}\left[\int_{-T/2}^{0} -t\,dt + \int_{0}^{T/2} t\,dt\right] = \frac{2A}{T^2}\left[-\frac{1}{2}t^2\Big|_{-T/2}^{0} + \frac{1}{2}t^2\Big|_{0}^{T/2}\right] = \frac{A}{T^2}\left[\frac{T^2}{4}+\frac{T^2}{4}\right] = \frac{A}{2}$$

$$a_n = \frac{4}{T}\int_{0}^{T/2}\frac{2A}{T}t\cos(n\omega t)\,dt = \frac{8A}{T^2}\int_{0}^{T/2} t\cos(n\omega t)\,dt = \frac{8A}{T^2}\left[\frac{t}{n\omega}\sin(n\omega t) + \frac{1}{(n\omega)^2}\cos(n\omega t)\right]_{0}^{T/2}$$

$$= \frac{8A}{T^2}\frac{1}{n\omega}\left[\underbrace{\cancel{\frac{T}{2}\sin\left(n\omega\frac{T}{2}\right)}}_{\text{Zero}} + \frac{1}{n\omega}\cos\left(n\omega\frac{T}{2}\right) - 0 - \frac{1}{n\omega}\right] = \frac{8A}{T^2}\frac{1}{n^2\omega^2}\left[\cos(n\pi) - 1\right]$$

$$= \frac{2A}{n^2\pi^2}\left[\cos(n\pi) - 1\right]$$

$n=1 \rightarrow a_n = -\frac{4A}{\pi^2}$

$2 \rightarrow a_n = 0$

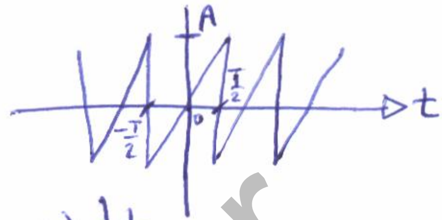
$3 \rightarrow a_n = -\frac{4A}{9\pi^2}$

$4 \rightarrow a_4 = 0$

$5 \rightarrow a_5 = -\frac{4A}{25\pi^2}$

$$\therefore a_n = \begin{cases} -\frac{4A}{n^2\pi^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

EX: Find the Fourier coefficients of the signal shown below.



solution: The function is odd, then $a_n = 0$ and $a_0 = 0$.

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega t)\,dt = \frac{8A}{T^2}\int_0^{T/2} t\sin(n\omega t)\,dt$$

$$= \frac{8A}{n\omega T^2}\left[-t\cos(n\omega t) + \frac{1}{n\omega}\sin(n\omega t)\right]_0^{T/2}$$

$$= \frac{8A}{n\omega T^2}\left[-\frac{T}{2}\cos\left(n\frac{2\pi}{T}\frac{T}{2}\right) + \frac{T}{n2\pi}\sin\left(n\frac{2\pi}{T}\frac{T}{2}\right) - 0 - 0\right]$$

$$= \frac{8A}{n\frac{2\pi}{T}T^2}\left[-\frac{T}{2}\cos(n\pi) + \underbrace{\cancel{\frac{T}{n2\pi}\sin(n\pi)}}_{\text{zero}}\right]$$

$$= \frac{-2A}{n\pi}\cos(n\pi) = -\frac{2A}{n\pi}(-1)^n$$

Thus: if the function is even; $f(t) = f(-t)$ then $b_n = \text{zero}$

if the function is odd, $f(t) = -f(-t)$ then $a_0 = 0$, $a_n = 0$

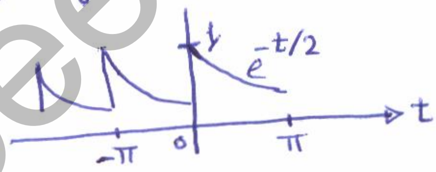
Exponential Fourier Series

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} D_n \, e^{jn\omega t}}$$

where

$$\boxed{D_n = \frac{1}{T} \int_T f(t) \, e^{-jn\omega t} \, dt}$$

EX. Find the exponential fourier series for the signal $f(t) = e^{-t/2}$ shown below in the interval $0 \leq t \leq \pi$



solution: $T = \pi \Rightarrow \omega = \frac{2\pi}{T} = 2$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} D_n \, e^{j2nt}$$

$$D_n = \frac{1}{T} \int_T f(t) \, e^{-j\omega t} \, dt = \frac{1}{\pi} \int_0^{\pi} e^{-t/2} \, e^{-j2nt} \, dt = \frac{1}{\pi} \int_0^{\pi} e^{-(\frac{1}{2} + j2n)t} \, dt$$

$$= \frac{-1}{\pi(\frac{1}{2} + j2n)} e^{-(\frac{1}{2} + j2n)t} \Big|_0^{\pi} = \frac{0.504}{1 + j4n}$$

$$\therefore f(t) = 0.504 \sum_{n=-\infty}^{\infty} \frac{1}{1 + j4n} e^{j2nt}$$

Fourier Series Spectra

* To plot Fourier spectra, express the Fourier series in the polar form:

$$D_n = |D_n| e^{j\theta_n} \quad \& \quad D_{-n} = |D_n| e^{-j\theta_n}$$

EX: from the Last example on page 63, we have

$$D_0 = 0.504$$

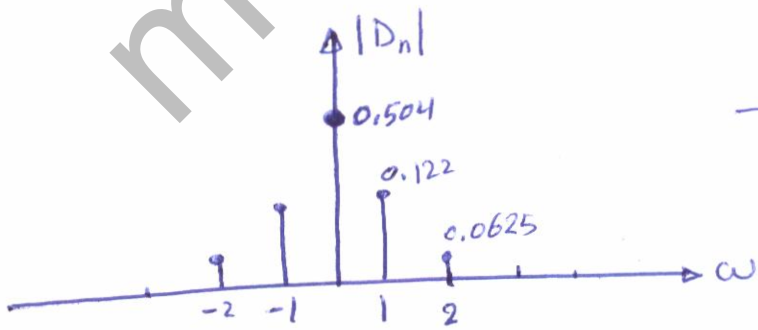
$$D_1 = \frac{0.504}{1+j4} = 0.122\, e^{-j(75.96)^\circ} \Longrightarrow |D_1| = 0.122, \quad \angle D_1 = -75.96^\circ$$

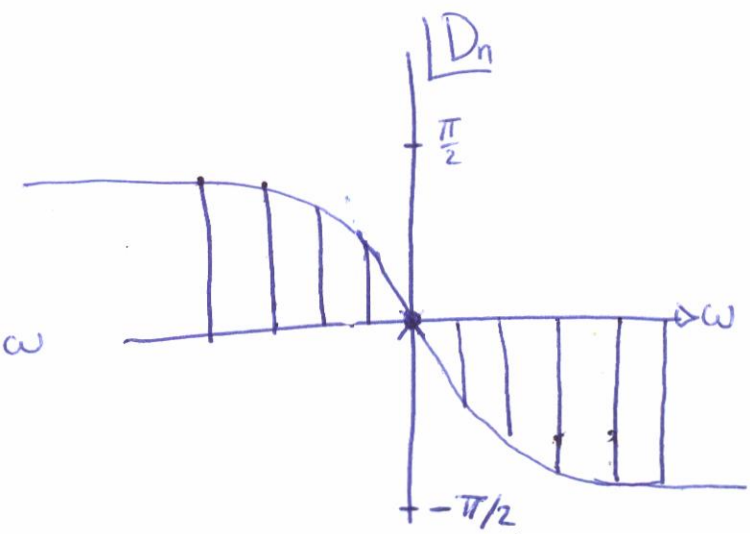
$$D_{-1} = \frac{0.504}{1-j4} = 0.122\, e^{j(75.96)^\circ} \Longrightarrow |D_{-1}| = 0.122, \quad \angle D_{-1} = 75.96^\circ$$

$$D_2 = \frac{0.504}{1+j8} = 0.0625\, e^{-j82.87^\circ} \longrightarrow |D_2| = 0.0625, \quad \angle D_2 = -82.87^\circ$$

$$D_{-2} = \frac{0.504}{1-j8} = 0.0625\, e^{j82.87^\circ} \longrightarrow |D_{-2}| = 0.0625, \quad \angle D_{-2} = 82.87^\circ$$

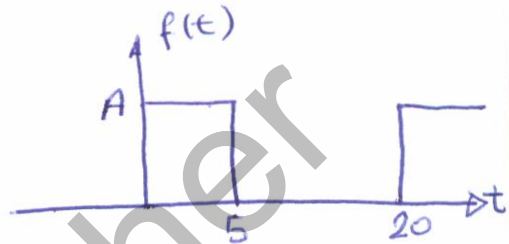
and so on





EX. Find the Fourier series coefficients for the signal $f(t) = A$ when $0 \le t < 5$ and zero in $5 \le t < 20$.

solution $f(t) = \begin{cases} A & 0 < t \le 5 \\ 0 & 5 < t \le 20 \end{cases}$



Period $T = 20$

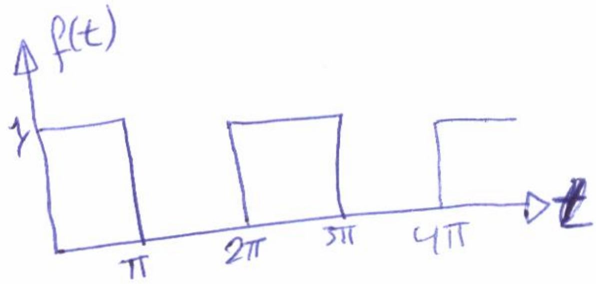
$$D_n = \frac{1}{T}\int_T f(t)\, e^{-jn\omega t}\, dt = \frac{1}{T}\int_0^{5} A\, e^{-jn\omega t}\, dt = \frac{A}{T}\int_0^5 e^{-j2\pi n f t}\, dt$$

$$= \frac{A}{T(-j2\pi \frac{n}{T})}\left[e^{-j2\pi\frac{n}{T}t}\right]_0^5 = \frac{-A}{j2\pi n}\left[e^{-j2\pi n f 5} - 1\right] = \frac{A}{j2\pi n}\left[1 - e^{-j2\pi n f 5}\right]$$

$$= \frac{A}{j2\pi n}\, e^{-j\pi n 5 f}\left[e^{j\pi n 5 f} - e^{-j\pi n 5 f}\right] = \frac{A\, e^{-jn\pi 5 f}}{n\pi}\sin\left(\frac{n\pi}{4}\right)$$

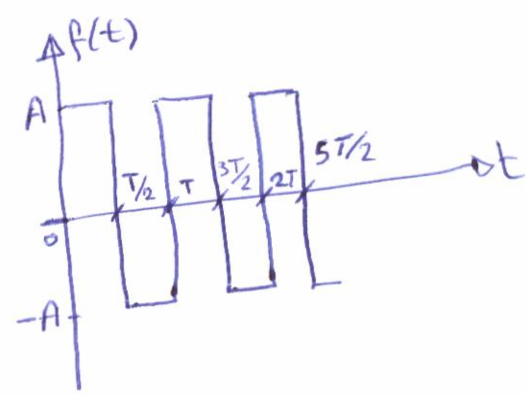
$$= \frac{8}{n\pi}\sin\left(\frac{n\pi}{4}\right) e^{-j\frac{n\pi}{4}}$$

Q1 / Determine the Fourier series for the waveform shown below



Ans: $f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right)$

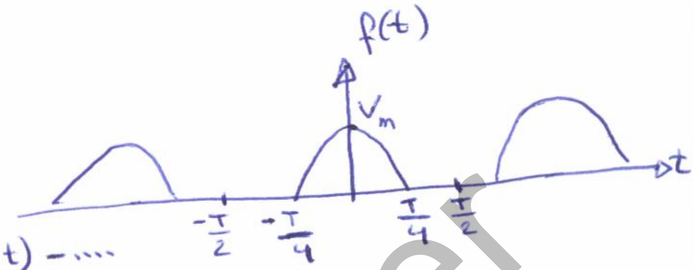
Q2 / Find the Fourier series of the waveform shown below



Ans: $f(t) = \frac{2A}{jn\pi}$ for $n = 1, 3, 5, 7 \ldots$

$f(t) = 0$ for $n = 2, 4, 6, 8, \ldots$

Q/ Obtain the trigonometric Forier series for the waveform shown below



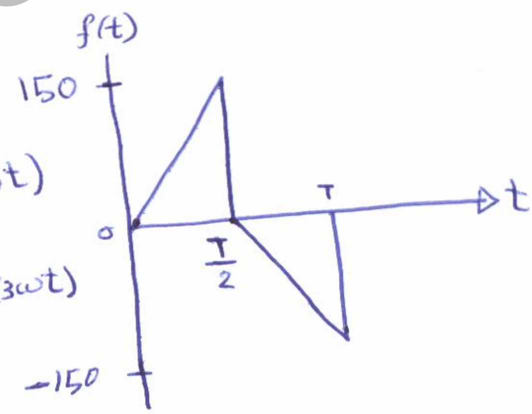
Ans: $f(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\cos(\omega t) + \frac{2V_m}{3\pi}\cos(2\omega t) - \frac{2V_m}{15\pi}\cos(4\omega t) + \frac{2V_m}{35\pi}\cos(6\omega t) - \cdots$

Q/ The output of a rectifier is $\begin{cases} V_m\cos(\omega t) & 0 \leq \omega t \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq \omega t \leq \frac{3\pi}{2} \\ V_m\cos\omega t & \frac{3\pi}{2} \leq \omega t \leq 2\pi \end{cases}$

find its Fourier series.

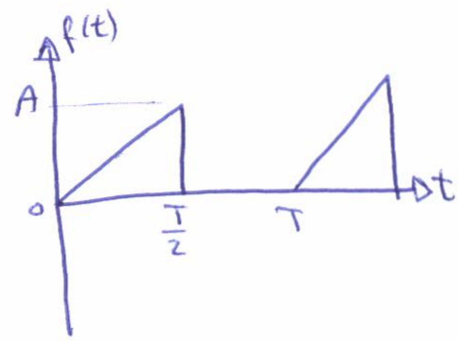
Ans: $f(t) = \frac{V_m}{2\pi} + \frac{V_m}{2\pi}\sin(\omega t) + \frac{V_m}{\pi}\sum_{n=2}^{\infty}\left(\frac{n}{n^2-1}\right)\sin(n\omega t)$

Q/ Determine the Fourier series of repetitive waveform of the figure below upto 7th harmonic when repetition time $T = 25\pi\,ms$.



Ans: $f(t) = \frac{600}{\pi^2}\cos(\omega t) - \frac{600}{3^2\pi^2}\cos(3\omega t) - \frac{600}{5^2\pi^2}\cos(5\omega t) - \frac{600}{7^2\pi^2}\cos(7\omega t) + \frac{300}{\pi}\sin(\omega t) + \frac{300}{3\pi}\sin(3\omega t) + \frac{300}{5\pi}\sin(5\omega t) + \frac{300}{7\pi}\sin(7\omega t)$

Q/ Obtain the trigonometric Fourier series for the signal shown below



Ans: $f(t) = \frac{A}{4} - \frac{2A}{\pi^2}\left[\cos(\omega t) + \frac{1}{3^2}\cos(3\omega t) + \frac{1}{5^2}\cos(5\omega t) + \cdots\right] + \frac{A}{\pi}\left[\sin(\omega t) - \frac{1}{2}\cos(2\omega t) + \frac{1}{3}\cos(3\omega t) - \frac{1}{4}\cos(4\omega t) + \cdots\right].$

Parseval's Power Theorem

The average power of a periodic signal is equal to the summation of the Fourier coefficients of the signal.

$$P_g = \frac{1}{T}\int_{T} |g(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |D_n|^2$$