❖ Complex Number

Since there is no real number (x), which satisfies the polynomial equation $x^2 + 1 = 0$,or similar equations, the set of complex numbers is introduced.

The complex number can be written in the form of $(a + bj)$, where (a, b) are real numbers and (j) is $(\sqrt{-1})$.

❖ Properties of Complex Number

- The two complex numbers $(a + bj)$ & $(c + dj)$ if $[a = c]$&$[b = d]$
- $(a + bj) \mp (c + dj) = (a \mp c) + (b \mp d)j$
- $(a + bj) * (c + dj) = (ac - bd) + (bc + ad)j$
- $\frac{(a+bj)}{(c+dj)} = \frac{(a+bj)*(c-dj)}{(c+dj)*(c-dj)} = \frac{(ac+bd)+(bc-ad)j}{(c^2+d^2)}$
- The complex conjugate of $(a \pm bj)$ is $(a \mp bj)$
- Absolute value of a complex number [z = $a + bj$] is $|z| = |a + bj|$
- = $\sqrt{a^2 + b^2}$
- The distance between two complex numbers $[z_1 = a + bj]$ & $[z_2 = c + dj]$ is $|z_1 - z_2| = \sqrt{(a-c)^2 + (b-d)^2}$
- The polar form of the complex number $[a + bj]$ is $(re^{j\theta})$ where $r = \sqrt{a^2 + b^2}$, and $\theta = tan^{-1}(\frac{b}{a})$
- $e^{j\theta} = \cos(\theta) + jsin(\theta)$
- $(j)^n = \begin{cases} -1\ when\ (n) is\ even \\ 1\ when\ (n) is\ odd \end{cases}$
- If $z_1 = r_1 e^{j\theta_1}$& $z_2 = r_2 e^{j\theta_2}$, then $z_1.z_2 = r_1 r_2 e^{j(\theta_1+\theta_2)}$
- If $z_1 = r_1 e^{j\theta_1}$& $z_2 = r_2 e^{j\theta_2}$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$
- If $z = re^{j\theta}$, then $z^n = r^n e^{jn\theta}$

- If $z = re^{j\theta}$, then $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j(\frac{\theta+2\pi k}{n})}$ or in rectangular coordinate if $z = a + bj$, then $z^{1/n} = [\ r\ (\cos\theta + J \sin\theta)\ ]^{1/n}$, where k = 0, ± 1 , ± 2, … … ..

Figure (1) shows how to convert from rectangular form to polar form, which gives:

$r = \sqrt{x^2 + y^2}y$

$\theta = \tan^{-1}(\frac{y}{x})$

$x = r\ \cos\theta$

$y = r\ \sin\theta$

Not that (θ) in radian

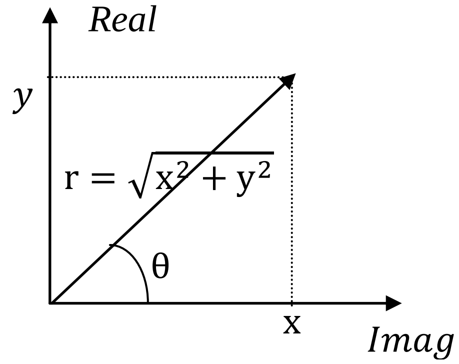


Fig (1)

- **<u>De . Moiver's Theorem</u>**

If $z_1 = x_2 + j\ y_1 = r_1\ (\cos\theta_1 + j\sin\theta_1) = r_1 e^{j\theta_1}$ &

$z_2 = x_2 + jy_2 = r_2\ (\cos\theta_2 + j\sin\theta_2) = r_2 e^{j\theta_2}$

Generally

$z_1 z_2 z_3 z_4 \dots . z_n = r_1 r_2 r_3 r_4 \dots . r_n[\cos(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \cdots \theta_n) +$
$j\sin(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \cdots \theta_n)]$

If $z_1 = z_2 = z_3 = z_4 = \cdots = z_n$, then

$z_1 z_2 z_3 z_4 \dots . z_n = = r^n[\cos(\theta n) + jsin(\theta n)$

Ex$_1$/ for ($z_1 = 3 + j4$), ($z_2 = -2 + j1$) & ($z_3 = j$) find and plot ($z_1 z_2 z_3$)

Sol:

From De. Moiver's Theorem

$z_1 z_2 z_3 = r_1 r_2 r_3[\cos(\theta_1 + \theta_2 + \theta_3) + jsin(\theta_1 + \theta_2 + \theta_3)$

$r_1 = \sqrt{(3)^2 + (4)^2} = 5, \theta_1 = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^o$

$r_2 = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}\ , \theta_2 = \tan^{-1}\left(\frac{-1}{2}\right) = -26.57^o$

$r_3 = \sqrt{(0)^2 + (1)^2} = 1\ , \theta_1 = \tan^{-1}\left(\frac{1}{0}\right) = 90^o$

$\rightarrow$

$z_1z_2z_3 = 5\sqrt{5}[cos(53.13^o - 26.57^o + 90^o) + jsin(53.13^o - 26.57^o + 90^o)$

$= 5\sqrt{5}[cos(116.6^o) + jsin(116.6^o)$

$\cong -5 + j10$

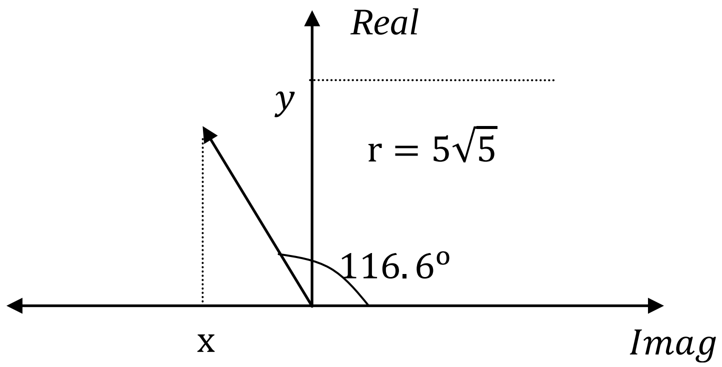


Fig (2)

Ex$_2$: find $\sqrt[\frac{1}{4}]{1 - j2}$

Sol:

Since $z = 1 - j2$ & $n = 4$, then $k = [0,1,2,3]$

$r = \sqrt{(1)^2 + (2)^2}$

$= \sqrt{5}$

$\theta = \tan^{-1}(\frac{-2}{1})$

$= 296.56^o$

$\cong 5.2\pi$

$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j(\frac{\theta + 2\pi k}{n})}$

$= (5)^{\frac{1}{8}} e^{j(\frac{5.2\pi + 2\pi k}{4})}$

If $k = 0 \rightarrow$

$z_0 = (5)^{\frac{1}{8}} e^{j(\frac{5.2\pi + 2\pi(0)}{4})}$

$= (5)^{\frac{1}{8}}\, e^{j\left(\frac{13}{10}\pi\right)}$

$= (5)^{\frac{1}{8}}\, [\cos\left(\frac{13}{10}\pi\right) + j\sin(\frac{13}{10}\pi)$

$= -0.72 - j0.99$

If $k = 1 \rightarrow$

$z_1 = (5)^{\frac{1}{8}}\, e^{j\left(\frac{5.2\pi + 2\pi(1)}{4}\right)}$

$= (5)^{\frac{1}{8}}\, e^{j\left(\frac{9}{5}\pi\right)}$

$= (5)^{\frac{1}{8}}\, [\cos\left(\frac{9}{5}\pi\right) + j\sin(\frac{9}{5}\pi)$

$= 0.989 - j0.72$

If $k = 2 \rightarrow$

$z_2 = (5)^{\frac{1}{8}}\, e^{j\left(\frac{5.2\pi + 2\pi(2)}{4}\right)}$

$= (5)^{\frac{1}{8}}\, e^{j\left(\frac{23}{10}\pi\right)}$

$= (5)^{\frac{1}{8}}\, [\cos\left(\frac{23}{10}\pi\right) + j\sin(\frac{23}{10}\pi)$

$= 0.72 + j0.989$

If $k = 3 \rightarrow$

$z_3 = (5)^{\frac{1}{8}}\, e^{j\left(\frac{5.2\pi + 2\pi(3)}{4}\right)}$

$= (5)^{\frac{1}{8}}\, e^{j\left(\frac{14}{5}\pi\right)}$

$= (5)^{\frac{1}{8}}\, [\cos\left(\frac{14}{5}\pi\right) + j\sin(\frac{14}{5}\pi)$

$= -0.989 + j0.72$

❖ Complex Variable

If to each of a set of complex numbers which a variable z may assume there corresponds one or more values of a variable (w), then w is called a function of the complex variable (z), written $w = f(z)$.

A function is single-valued if for each value of (z) there corresponds only one value of w; otherwise, it is multiple valued or many-valued. In general, we can write $[w = f(z) = u(x,y) + iv(x,y)]$, where (u) and (v) are real functions of (x) and (y).

Ex$_3$/ $w = z^2 = (x + jy)^2 = x^2 - y^2 + 2ixy = u + iv$ so that $(x,y) = x^2 - y^2$, $v(x,y) = 2xy$.

These are called the real and imaginary parts of $w = z^2$ respectively.

Ex$_4$/ find $(v \,\&\, v)$ in term of $(x \,\&\, y)$if $g = u^2 - j\tan^{-1}(v)$, $z = \ln(x) - jy$, and $g = z - 2$

Sol:

Since $g = z - 2 \rightarrow g = \ln(x) - jy - 2$

$\rightarrow u^2 - j\tan^{-1}(v) = \ln(x) - jy - 2$

$\therefore\ u^2 = \ln(x) - 2 \rightarrow u = \sqrt{\ln(x) - 2}$

$\tan^{-1}(v) = y \rightarrow v = \tan(y)$

Ex$_5$/ write the function $w = z^2 + e^z$ in the form w= m(x,y)+JU(x,y)

Sol: setting z= x+Jy

$z = f(z) = (x + jy)^2 + 2(x + jy)$

$W = x^2 - y^2 + J2xy + 2x + 2Jy$

$W = (x^2 - y^2 + 2x) + J(2xy + 2y)$

$W = (x^2 - y^2 + 2x) + 2J(xy + y)$

$u = x^2 + 2X - y^2$ &$v = 2(xy + y)$

Ex$_6$/ graph the function $[w = z^2]$ in (uv) plane?

Sol: since $z = x + jy$

$\therefore z^2 = x^2 - y^2 + 2jxy$

$w = u(x, y) + jv(x, y)$

Where $u = x^2 - y^2$, $v = 2xy$

z		w	
x	y	u	v
1	0	1	0
$\sqrt{2}$	$\sqrt{2}$	0	4
$1/\sqrt{2}$	$1/\sqrt{2}$	0	1
0	1	-1	0
-1	0	1	0

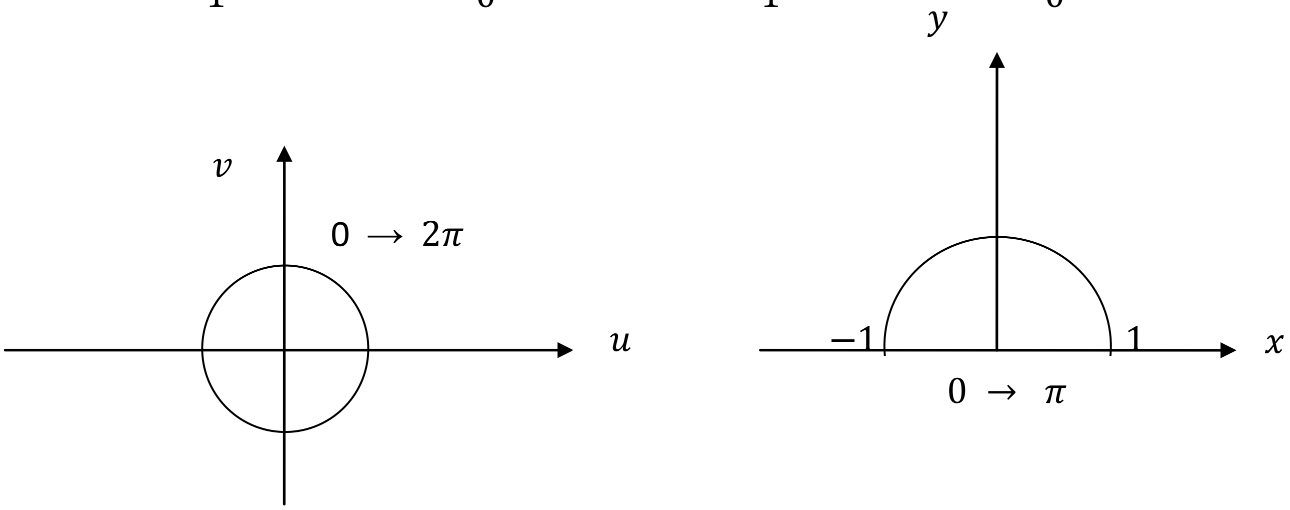


Fig (3)

❖ Derivative of Complex Variable

If $w = f(z)$, $w = u + jv$, and $z = x + jy$, then

$$\frac{dw}{dx} = \frac{du}{dx} + j\frac{dv}{dx} \quad \dots\dots\dots(1)$$

$$\frac{dw}{dy} = \frac{du}{dy} + j\frac{dv}{dy} \quad \dots\dots\dots(2)$$

❖ Cauchy- Riemann Conditions

The following two conditions are called Cauchy- Riemann Conditions, and these two conditions are considered two now if the function of complex variables are analytic or not.

$$\frac{du}{dx} = \frac{dv}{dy} \qquad \dots\dots\dots (3)$$

$$\frac{du}{dy} = -\frac{dv}{dx} \qquad \dots\dots\dots (4)$$

Proof:

Since $w = f(z)$

$w = u + jv$, and $z = x + jy$

$$\frac{dz}{dx} = 1, \frac{dz}{dy} = j$$

$$\frac{dw}{dx} = \frac{df}{dz}\frac{dz}{dx} = \frac{df}{dz} \qquad \dots\dots\dots (5)$$

From equations (1&5), it can be found that

$$\frac{df}{dz} = \frac{du}{dx} + j\frac{dv}{dx} \qquad \dots\dots\dots (6)$$

In addition:

$$\frac{dw}{dy} = \frac{df}{dz}\frac{dz}{dy} = j\frac{df}{dz} \qquad \dots\dots\dots (7)$$

$$j\frac{df}{dz} = \frac{du}{dy} + j\frac{dv}{dy} \qquad \dots\dots\dots (8)$$

Substitutes equation(6) in (8)

$$j\left(\frac{du}{dx} + j\frac{dv}{dx}\right) = \frac{du}{dy} + j\frac{dv}{dy}$$

$$j\frac{du}{dx} - \frac{dv}{dx} = \frac{du}{dy} + j\frac{dv}{dy} \rightarrow$$

$$\frac{du}{dy} = -\frac{dv}{dx} \quad \& \quad \frac{du}{dx} = \frac{dv}{dy} \qquad \dots\dots\dots (9)$$

❖ **Analyticity**

The most important condition that must be considered to decide that the function is analytic or not is the Cauchy-Riemann conditions, the function must be regular within a region (R) which means that all points (z_o) in the region (R)is single valued in this region and have a unique finite value.

Ex$_7$/ is the following function is analytic or not?

$f(z) = \cos(z)$ where $z = x + jy$

$$f(z) = \frac{e^{jz} + e^{-jz}}{2}$$

$$= \frac{e^{j(x+jy)} + e^{-j(x+jy)}}{2}$$

$$= \frac{1}{2}(e^{jx}e^{-y} + e^{-jx}e^{y})$$

$$= \frac{1}{2}\left[e^{-y}\big(\cos(x) + jsin(x)\big) + e^{y}\big(\cos(x) - jsin(x)\big)\right]$$

$$= \frac{1}{2}\left[e^{-y}\cos(x) + e^{y}\cos(x) + je^{-y}sin(x) - je^{y}sin(x)\right]$$

$$= \frac{1}{2}\left[(e^{y} + e^{-y})\cos(x) + j(e^{-y} - e^{y})sin(x)\right]$$

This mean

$$u = \frac{1}{2}(e^{y} + e^{-y})\cos(x)$$

$$v = \frac{1}{2}(e^{-y} - e^{y})\sin(x)$$

$$\frac{du}{dx} = \frac{-1}{2}(e^{y} + e^{-y})\sin(x)$$

$$\frac{dv}{dy} = \frac{1}{2}(-e^{-y} - e^{y})\sin(x) = \frac{-1}{2}(e^{-y} + e^{-y})\sin(x)$$

$$\frac{du}{dy} = \frac{1}{2}(e^{y} - e^{-y})\cos(x)$$

$$\frac{dv}{dx} = \frac{-1}{2}(\mathrm{e}^{-\mathrm{y}} - \mathrm{e}^{\mathrm{y}})\cos(x) = \frac{1}{2}(\mathrm{e}^{\mathrm{y}} - \mathrm{e}^{-\mathrm{y}})\cos(x)$$

Since

$$\frac{du}{dx} = \frac{dv}{dy} \;\; \& \;\; \frac{du}{dy} = -\frac{dv}{dx}$$

∴ The function $f(z) = \cos(\mathrm{z})$ is analytic function.

Ex$_8$/ If $(Z = x + jy)$, is the function $(G = \frac{2}{Z})$ analytic or not?

Sol:

$$G = \frac{2}{Z}$$

$$= \frac{2}{x + jy} \cdot \frac{x - jy}{x - jy}$$

$$= \frac{2x - j2y}{x^2 + y^2} \rightarrow$$

$$u = \frac{2x}{x^2 + y^2} \quad \& \; v = \frac{2y}{x^2 + y^2}$$

$$\frac{du}{dx} = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$\frac{dv}{dy} = \frac{-2(x^2 + y^2) + 2y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{-2x^2 - 2y^2 + 2y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

And in the same way

$$\frac{du}{dv}=\frac{-2x(2y)}{(x^2+y^2)^2}$$

$$=\frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{dv}{dx}=\frac{-2y(2x)}{(x^2+y^2)^2}$$

$$=\frac{-4yx}{(x^2+y^2)^2}$$

$$\therefore\ \frac{du}{dx}=\frac{dv}{dy}\ \ \&\ \frac{du}{dy}=-\frac{dv}{dx}$$

This means that the Cauchy-Riemann conditions are satisfied but not only for all points, for example the point (0, 0)

At $y=0\rightarrow u=\frac{1}{x}$ and $\frac{du}{dx}=\frac{-1}{x^2}$

Similarly at $x=0\rightarrow v=\frac{-1}{y}$ and $\frac{dv}{dy}=\frac{-1}{y^2}$

$$\frac{du}{dx}\neq\frac{dv}{dy}\ \ \&\ \frac{du}{dy}=-\frac{dv}{dx}=0$$

∴The Cauchy-Riemann conditions are not fully satisfied, since the function ($G=\frac{2}{Z}$) is not analytic at point ($Z=0$)

❖ **<u>Some Important functions</u>**

1- Harmonic

Re call Cauchy-Riemann conditions:

$$\frac{du}{dx}=\frac{dv}{dy}\qquad \ldots\ldots\ldots(3)$$

$$\frac{du}{dy}=-\frac{dv}{dx}\qquad \ldots\ldots\ldots(4)$$

Derive the two sides of equation (3) with respect to (x) and the two sides of equation (4)

$$\frac{d^2u}{dx^2} = \frac{d^2v}{dydx} \qquad \dots\dots\dots(10)$$

$$\frac{d^2u}{dy^2} = -\frac{d^2v}{dxdy} \qquad \dots\dots\dots(11)$$

From (10) & (11) it can be found that

$$\frac{d^2u}{dx^2} = -\frac{d^2u}{dy^2}$$

Ex$_9$/ if $(z = x + jy)$ is the function $(w = z + 1)$ harmonic or not?

Sol:

Since $= u + jv$, then $u + jv = x + jy + 1$

$\rightarrow u = x + 1$ & $v = y$

$$\frac{du}{dx} = 1 \rightarrow \frac{d^2u}{dx^2} = 0$$

$$\frac{du}{dy} = 0 \rightarrow \frac{d^2u}{dy^2} = 0$$

∴ the function is harmonic.

Hw$_1$: if (z) is acomplex number and $[G = e^{(z-1)}]$, is (G) function is harmonic or not.

Hw$_2$: and find prove that $(w = e^{-x}(xsin(y) - ycos(y))$ is a harmonic function, and find each of $(u\ \&v)$ if $(w = u + jv)$

Hw$_3$: is $(w = \sqrt[3]{z^2 - 2}$) harmonic function or not, if $(z = x + jy)$.

2- Elementary Function

Such as exponential function $(f(z) = e^z)$ which is analytic function, to prove that:

If $z = x + jy \rightarrow e^z = e^{(x+jy)} = e^x\big(\cos(y) + jsin(y)\big)$

$\rightarrow u = e^{x}\cos(y) \therefore \frac{du}{dx} = e^{x}\cos(y) \;\&\; \frac{du}{dy} = -e^{x}\sin(y)$

$v = e^{x}\, sin(y) \;\therefore \frac{dv}{dx} = e^{x}\sin(y) \;\&\; \frac{dv}{dy} = e^{x}\cos(y)$

❖ Complex Integration

In this section, two types of integration will be illustrated

1- Line integral

Which is denoted by $\int_{c} f(z)dz$

Or $\int_{a}^{b} f(z)dz$

For $f(z) = u + jv \;\&\; z = x + jy \rightarrow dz = dx + jdy$

$\rightarrow \int_{c} f(z)dz = \int_{c} (u + jv)(dx + jdv)$

$$\int_{c} f(z)dz = \int_{c} [(udx - vdy) + j(udy + vdx)]$$

Ex$_{10}$/ find ($\int_{c} e^{z}\, dz$) from $[z = 0$ to $z = 2 + j2]$

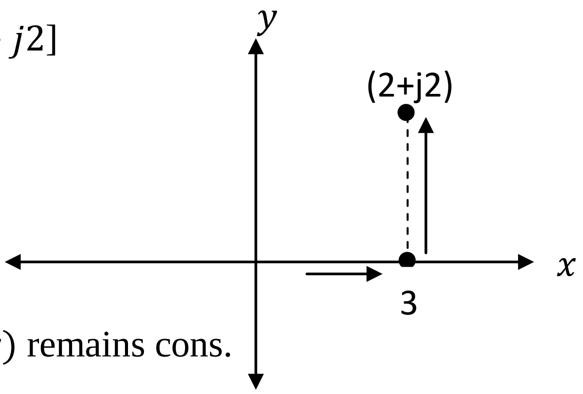


Fig (4)

Sol:

First path:

From $z = 0 + j0$ to $z = 3 + j0$

In this path (x) changes from (0) to (2) & (y) remains cons.

$e^{z} = e^{x+jy}$

$= e^{x}[\cos(y) + jsin(y)]$

$= e^{x}\cos(y) + je^{x}\sin(y) \rightarrow u = e^{x}\cos(y) \;\&\; v = e^{x}\sin(y)$

$x = (0\; to\; 2) \;\&\; y = 0 \rightarrow$

$I = \int_0^2 [e^x \cos(0)\, dx - e^x \sin(0)\, dy] + j \int_0^2 [e^x \cos(0) dy + e^x \sin(0) dx]$

$= \int_0^2 e^x\, dx$

$= [e^x]_0^2$

$= (e^2 - 1)$

For the second path:

$y = (0\ to\ 2)\ \&\ x = 2 \rightarrow$

$I = \int_0^2 [e^2 \cos(y)\, dx - e^2 \sin(y)\, dy] + j \int_0^2 [e^2 \cos(y) dy + e^2 \sin(y) dx]$

$= -\int_0^2 e^2 \sin(y)\, dy + j \int_0^2 e^2 \cos(y) dy$

$= e^2 [\cos(y)]_0^2 + j e^2 [\sin(y)]_0^2$

$= e^2 \cos(2) - e^2 \cos(0) + j e^2 \sin(2) - e^2 \sin(0)$

$= \ e^2 [\cos(2) + j \sin(2)] - e^2$

$= \ e^2 \left(e^{j2} - 1 \right)$

$\therefore I_t = \ (e^2 - 1) + \ e^2 \left(e^{j2} - 1 \right)$

$= e^{j4} - 1$

Ex11/ find $\int_c (z^2 - 2) dz$, if (z)changes from $(-1 + j1)$ to $(2 + j1)$ & $(y = 2x)$

Sol:

$f(z) = z^2 - 2$

$= (x^2 - y^2 + 2jxy) \rightarrow u = x^2 - y^2\ \&\ v = \ 2xy$

$\rightarrow I = \int_c [(x^2 - y^2) dx - \ 2xy dy] + j \ \int_c [(x^2 - y^2) dy + \ 2xy dx]$

Since $(y = 2x \ \rightarrow dy = 2dx)$, then

$$I = \int_{-1}^{2}[(x^2 - 4x^2)dx - \ 8x^2dx] + j\int_{-1}^{2}[2(x^2 - 4x^2)dx + \ 4x^2dx]$$

$= \frac{11}{3}\ [x^3]_{-1}^{2} + j\ \frac{2}{3}\ [x^3]_{-1}^{2}$

$= \frac{11}{3}\ [8+1] + j\frac{2}{3}\ [8+1]$

$= 33 + j\ 6$

❖ **Change of Variable**

Let $z = g(\alpha)$ be a continuous function of a complex variable $\alpha = u + jv$ suppose that the curve (c)in the (z) plane corresponding to curve $(\bar{c})$ with the (α)plane & that the derivative $\bar{g}(\alpha)$ is continuo's on $(\bar{c})$ then :

$\int_c f(z)dz = \int_c f\{g(\alpha)\}\bar{g}(\alpha)d\alpha$

Ex$_{12}$:- evaluate $\int_c z\,dz$ from $z = 0$ to $z = 4 + j2$ along the curve c given by $z = t^2 + jt$?

Sol:- here $t = \alpha$

$g(t) = t^2 + jt \Rightarrow \bar{g}(t) = 2t + j$

when $z = 0$

Since $z = t^2 + jt \Rightarrow t = 0$

$Z = 4 + 2j$

$\therefore\ 4 + 2j = t^2 + jt \Rightarrow t = 2$

$\therefore \int_c zdz = \int_0^2 (t^2 + \mathrm{j}t)(2t + j)dt$

$= 6 + 8j$

Hw$_4$/ evalute $\int_c z\,dz$ from $z = 1 + j3$ to $z = 2 + j4$ along the curve (c) given by $z = e^{-ju}$

Remark$_1$: when the direction of the counter integration is changed the the sign of integration changes too.

Remark$_2$: Simple closed curve, simple & multiply connected region.

A curve is called a simple closed curve if does not cross it self figure (5.1) is a simple closed curve while figure (5.2) is not simple closed & is known as multiple curve .

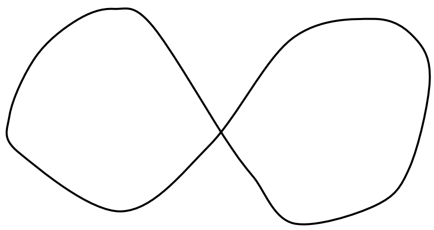


Fig 5.2

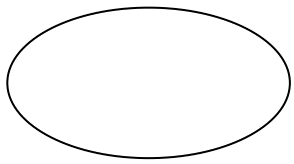


Fig 5.1

Remark$_3$: Aregion is called simple connected if every closed curve in the region enclosed point of the region only . aregion which is not simple connected is called multiply connected for example :The region between two concentric circuit $r_1 \le |z - z_0| \le r_2$ as shown in figure (6) is an example of amultiple connected region .

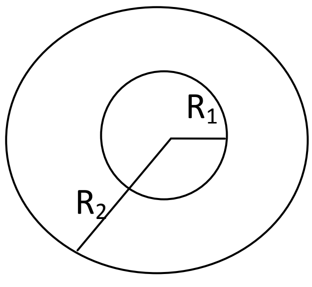


Fig .6

[a simple connected region is one which has non hole]. A region with one hole is called double connected region with two holes is called triply connected(x) so on.

❖ Cauchy's theorem of integral

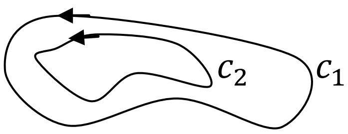
This theorem states that, when $f\ (z)$ is analytic & $\bar{f}(z)$ is continuous inside & on simple closed curve, then

$$\oint_{c} f(z)dz = 0 \ldots\ldots\ldots (12)$$

This integration is called a contour integration.

If $f(z)$ is analytic in the doubly connected region bounded by the curve $c_1 \& c_2$ as illustrated in figure (7) then:

$$\oint_{c_1} f(z)dz = \oint_{c_2} f(z)dz \ \ldots\ldots\ldots (13)$$



Fig(7)

Ex$_{13}$ / evaluate [$\oint_{c_1} \frac{dz}{z-a}$], where (c) is any simple closed curve &$(z = a)$ is

1- outside the curve (c)
2- inside the curve (c)

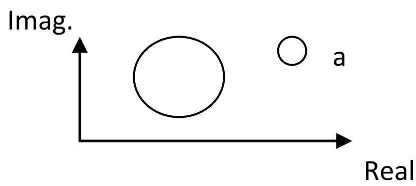


Fig (8)

Sol:

1- outside the curve (c)

$\oint_{c} \frac{dz}{z-a}$ =0

2- inside the curve (c)

Let be a Circular of radius (ϵ_o), with center at $Z = a$ so that (Γ) is inside (c) , recall Cauchy's theorem for multiply connected Region

$\oint_{c_1} f(z)dz = \oint_{c_2} f(z)dz$

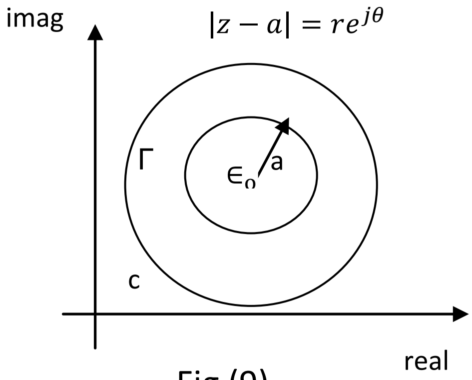


Fig (9)

$\oint_c \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a}$

$|z-a| = \epsilon_o$

$z-a = \epsilon_o e^{j\theta}$, $(z = e^{j\theta} + a)$

diff . both side

$dz = \mathrm{j}\, \epsilon_o\, e^{j\theta}\, d\theta$

For counter integration around the circuit of radius (ϵ_o) , lower limit $(\theta = 0)$ & upper limit $(\theta = 2\pi)$

$\therefore \oint_\Gamma \frac{dz}{z-a} = \oint_\Gamma \frac{j\epsilon e^{j\theta} d\theta}{\epsilon e^{j\theta}+a-a} = \mathrm{j} \oint_0^{2\pi} d\theta = 2\pi\, \mathrm{j} = \oint_c \frac{dz}{(z-a)}$

❖ Cauchy's integral formula

If $f(z)$ analyticall inside & on asimple closed curve (c & z_o) is any point inside (c) then :

$f(z_o) = \frac{1}{2\pi \mathrm{j}} \oint_c \frac{f(z)}{z-z_0} dz$ ……… (14)

or $\oint_c \frac{f(z)}{z-z_0} dz = 2\pi j\, f(z_o)$ ……… (15)

The derivatives of (n) order for $f(z)$ at $(z = z_o)$ is given by

$f^n(z_o) = \frac{n!}{2\pi j} \oint_c \frac{f(z)}{(z-z_o)^{n+1}} dz$ ……… (16)

where n =1,2,3,…….order of derivative .

Ex$_{14}$: evaluate $\oint_c \frac{e^{-z}}{(z-3)(z-2)} dz$, where (c) is the circle, |z|= 3?

Sol:

Decompose the denominator

$\frac{1}{(z-3)(z-2)} = \frac{B}{z-3} + \frac{A}{z-2}$

$\rightarrow$

$$\frac{Bz-2B+Az-3A}{(z-3)(z-2)}=\frac{1}{(z-3)(z-2)}$$

$\rightarrow Bz-2B+Az-3A=1$

$\rightarrow \quad B+A=0 \quad \dots\dots\dots(17)$

$\underline{2B-3A=1} \quad \dots\dots\dots(18)$

Multiply equation (17) by (3) and adding the two equations (17),(18)

$\rightarrow \quad B+3A=0 \quad \dots\dots\dots(19)$

$\underline{2B-3A=1} \quad \dots\dots\dots(20)$

$3B=1 \rightarrow B=\frac{1}{3}$ substitutes in equation (17) then

$\frac{1}{3}+A=0 \rightarrow A=-\frac{1}{3}$

$\therefore \oint_c \frac{e^{-z}}{(z-3)(z-2)}dz=\frac{1}{3}\left(\oint_c \frac{e^{-z}}{(z-3)}dz-\oint_c \frac{e^{-z}}{(z-2)}dz\right)$

According to the Caushys integral formula

$\oint_c \frac{e^{-z}}{(z-3)}dz=2\pi\, j\, f(3)$

$=2\pi\, j\, e^{-3}$

& $\oint_c \frac{e^{-z}}{(z-2)}dz=2\pi\, j\, e^{-2}$

This means that $\oint_c \frac{e^{-z}}{(z-3)(z-2)}dz=\frac{1}{3}[2\pi\, j\, e^{-3}+2\pi\, j\, e^{-2}]$

Ex$_{15}$: evaluate $\oint_c \frac{\cos(2z)}{(z+\pi)^3}$ where (c) is the circle |z|=3?

Sol:

By Cauchy's formula

$f^{n}(z_o)=\frac{n!}{2\pi j}\oint_c \frac{f(z)}{(z-z_o)^{n+1}}$ dz

$n+1 = 3 \Longrightarrow n=2$

$f(z)=\cos(2z) \rightarrow \bar{f}(z)= -2\sin(2z), \bar{\bar{f}}(z)=-4\cos(2z)$

$\bar{\bar{f}}(z)=-4\cos(2z) \rightarrow \bar{\bar{f}}(\pi)= -4\cos(2\pi)$

$\bar{\bar{\bar{f}}}(\pi) = -4$

$\rightarrow$

$\bar{\bar{\bar{f}}}(\pi) = \frac{2!}{2\pi j} \oint_c \frac{f(z)}{(z-z_o)^{n+1}}$ dz

$\therefore \oint_c \frac{f(z)}{(z-z_o)^{n+1}} dz = \frac{2}{2!} \pi j * -4$

$= -4\pi j$

Hw5: Show that $\frac{1}{2\pi j} \oint_c \frac{e^{zx}}{z^2+1} dz = \sin(x)$, where (c) is a circle |z|=3

Ex16/ if (c) is a circle [$|z+2|$ =3], evaluate $\oint_c \frac{z^2}{z^2+1} dz$

Sol:

Since $|z+2|$ is a circle that centred at $(-2 - j0)$

and has raduis $= 3$

$z^2 + 1 = 0$

$z^2 = -1 \rightarrow z = \pm j$

The two points $[z = j,\ z = -j]$ are located inside the circle therefore

1-) $\oint_c \frac{z^2}{z^2+1} dz = 2\pi j\ f(-j)$

$= 2\pi j\ (-j)^2$

$= -2\ \pi j$

2-) $\oint_c \frac{z^2}{z^2+1} dz = 2\pi j\ f(j)$

$= 2\pi j\ (j)^2$

$= -2\ \pi j$

$$\therefore \oint_c \frac{z^2}{z^2+1} dz = -2\ \pi j - 2\ \pi j$$

$= -4\pi j$

Remark4: if $f(z)$ is analytic at any point except at $(z = z_o)$, then this point is called pole which is divided into [simple pole & pole of order (n)]

Remark5: when the analytic function has poles, then such function have a singularity at these poles.

Remark$_6$: if the pole of analytic function can be removed by taking $[\lim_{z\to z_o} f(z)]$, then this singularity is called removable singularity.

Remark$_7$: if the pole cannot be removed, then this singularity is called (essential singularity)

Remark$_8$: the analytic function is called entire function if this function is analytic at all points except at $(\pm\infty)$

Ex$_{17}$/ classify each of the following functions

1-) $f(z) = \frac{z-3}{(z-1)(z+3)}$ 2-) $f(z) = \frac{e^{-z}}{(z-8)^4}$ 3-) $f(z) = \frac{1}{e^{-\frac{1}{z}}}$

4-) $f(z) = \cos(z)$

Sol:

1- For the first function $f(z) = \frac{z-3}{(z-1)(z+3)}$

This function is analytic with simple poles $(1, -3)$ and has removable singularity

$\lim_{z\to 1} \frac{z-3}{(z-1)(z+3)}$ Apply H.R $\rightarrow \lim_{z\to 1} \frac{-3}{(-1)(3)} = 1$

Moreover, $\lim_{z\to -3} \frac{z-3}{(z-1)(z+3)}$ Apply H.R $\rightarrow \lim_{z\to 1} \frac{-3}{(-1)(3)} = 1$

2- For the function $f(z) = \frac{e^{-z}}{(z-8)^4}$

This function is analytic with pole (8) of order (4) and has removable singularity

$\lim_{z\to 8} \frac{e^{-z}}{(z-8)^4}$ Apply H.R $\rightarrow \lim_{z\to 8} \frac{-e^{-z}}{4(z-8)^3}$ Apply H.R $\rightarrow \lim_{z\to 8} \frac{e^{-z}}{12(z-8)^2}$

Apply H. R $\rightarrow \lim_{z\to 8} \frac{-e^{-z}}{24(z-8)^1}$ Apply H. R $\lim_{z\to 8} \frac{e^{-z}}{24} = \frac{e^{-8}}{24}$

3- For the function $f(z) = \frac{1}{e^{-\frac{1}{z}}} = e^{\frac{1}{z}}$

This function is analytic with pole (0) and has essential singularity

4- For the function $f(z) = \cos(z)$

This function is analytic on all the field except at ($\pm\infty$) then it is called entire function.

❖ <u>Residue Theorem</u>

The residue of a simple pole is given by

$$R\,(f, z_0) = R(z_0) = \lim_{z\to z_o}(z - z_0)f(z) \dots \dots \dots (21)$$

And the residue of a pole with order (*n*) is given by

$$R\,(f, z_0) = R(z_0) = \lim_{z\to z_o} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[(z - z_0)^n f(z)] \dots \dots \dots (22)$$

Ex$_{18}$/ evaluate the residue of ($f(z) = \frac{e^{-z}}{(z-2)^2}$

Sol: since the pole (2) has order (*n*) then the residue is given by

$$R\,(2) = \lim_{z\to 2} \frac{1}{(2-1)!}\frac{d^1}{dz^1}[(z-2)^2 \frac{e^{-z}}{(z-2)^2}] \rightarrow$$

$$R\,(2) = \frac{1}{(2-1)!}\lim_{z\to 2}\frac{d^1}{dz^1}[\,e^{-z}]$$

$$= \lim_{z\to 2}(\,-e^{-z}) \rightarrow R\,(2) = -e^{-2}$$

Remark$_9$: if $f(z)$ is analytic on a simple curve (c) except at an arbitrary points (a, b, c, …)that lies on or inside the curve (c), then it has a residue

Remark$_{10}$: before solve the integration using residue theorm it is important that to draw the curve (c) to know which poles inside the curve and which not.

$$\oint_c f(z)\,dz = 2\pi j[R(a) + R(b) + R(c) + \ldots] \ldots\ldots\ldots (23)$$

Or $\oint_c f(z)\,dz = 2\pi j \sum_{i=0}^{n} R(z_i) \ldots\ldots\ldots (24)$

Ex$_{19}$/ find $\oint \frac{\cos(z)}{(z-1)(z-3)}dz$ using residue theorem on the curve $[z = 2]$

Sol:

1- Draw the simple curve $[z = 2]$
2- Find the poles $[z = 1, z = 3]$
3- Find the residue of the poles that lies

Inside the curve $[z = 2]$

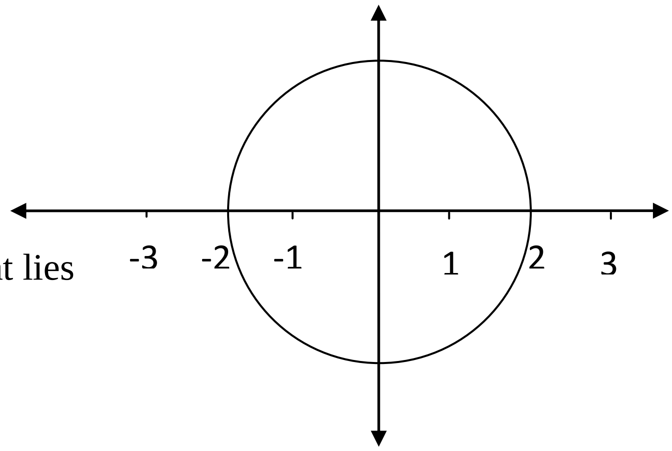


Fig (10)

Only $[R(1)]$ is considered

$$R(1) = \lim_{z\to1}(z-1)\frac{\cos(z)}{(z-1)(z-3)} \rightarrow$$

$$R(1) = \lim_{z\to1}\frac{\cos(z)}{(z-3)}$$

$$= \frac{\cos(1)}{(1-3)} \rightarrow R(1) = \frac{\cos(1)}{(-2)} \rightarrow \oint \frac{\cos(z)}{(z-1)(z-3)}dz = 2\pi j * \left[\frac{-1}{2}\cos(1)\right] \rightarrow$$

$$\oint \frac{\cos(z)}{(z-1)(z-3)}dz = -2\pi j\cos(1)$$

HW$_6$: find $\oint \frac{z}{z^2+z+\frac{5}{4}}\, dz$ on the circle ($|z-2-j3|$ using residue method.

Ex$_{20}$/ evaluate $\oint \frac{\sqrt{z}dz}{z^2+\sqrt{2}z-4}$ around the rectangle $[-2-j1, 2+j1, -2+j1$ and $2-j1]$

Sol:

1- Draw the rectangle
2- Solve the equation of

The denominator

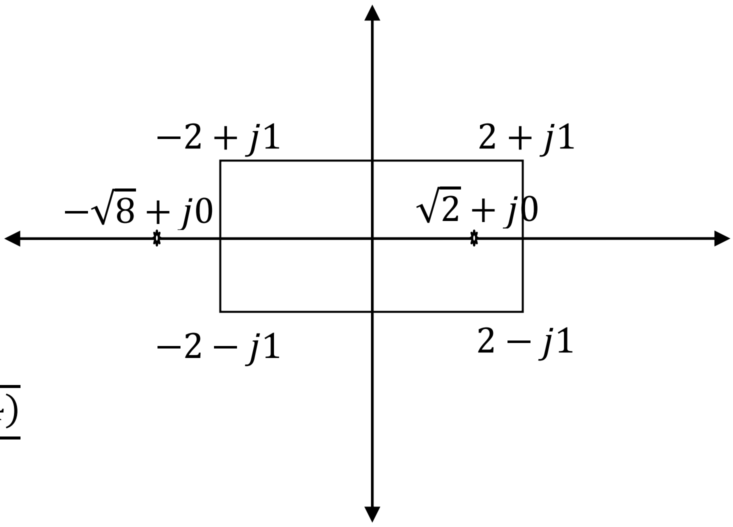


Fig (11)

$$z = \frac{-\sqrt{2} \pm \sqrt{2-4*1*(-4)}}{2*1}$$

$$z = \frac{-\sqrt{2}}{2} \pm \frac{\sqrt{18}}{2} \rightarrow z = \frac{-\sqrt{2} \pm 3\sqrt{2}}{2} \rightarrow z_1 = -2\sqrt{2}, z_2 = \sqrt{2}$$

Or $z_1 = -\sqrt{8}, z_2 = \sqrt{2} \therefore z^2 + \sqrt{2}z - 4 = \left(z+\sqrt{8}\right)\left(z-\sqrt{2}\right)$

$\rightarrow \oint \frac{\sqrt{z}}{z^2+\sqrt{2}z-4} = \oint \frac{\sqrt{z}}{(z+\sqrt{8})(z-\sqrt{2})}$

The first pole = $\sqrt{2}+j0$ & the second pole = $-\sqrt{8}+j0$

From figure (11), it can be found that the first pole inside the rectangle, while the second pole is outside the rectangle, therefore the just first pole is considered.

$$R\left(\sqrt{2}\right) = \lim_{z\rightarrow\sqrt{2}} \left(z-\sqrt{2}\right)\frac{\sqrt{z}}{\left(z-\sqrt{2}\right)\left(z+\sqrt{8}\right)} \rightarrow R\left(\sqrt{2}\right) = \lim_{z\rightarrow\sqrt{2}} \frac{\sqrt{z}}{\left(z+\sqrt{8}\right)}$$

$=\frac{\sqrt{2}}{(\sqrt{2}+2\sqrt{2})} \rightarrow R(\sqrt{2}) = \frac{1}{3}\longrightarrow$

$$\oint \frac{\sqrt{z}dz}{z^2+\sqrt{2}z-4} = 2\pi j * \frac{1}{3}$$

$=\frac{2}{3}\pi j$

<u>Remark$_{11}$:</u> when $f(z)$ is analytic and does not has a direc singular point such as $(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ is any analytic function, while $Q(z)$ is analytic function at all field except at some poits such as $[\sin(z), \cos(z), \ldots.]$then the residue is calculated as:

$$R(z_o) = \frac{P(z_o)}{\bar{Q}(z_o)} \quad \ldots \ldots \ldots (25)$$

Ex$_{21}$/ find the residue of the following functions:

1- $f(z) = tan(z)$
2- $f(z) = \sec(z)$

Sol:

1- $f(z) = tan(z)$

Since $tan(z) = \frac{\sin(z)}{\cos(z)}$

$\cos(z) = 0 \; at \; z = \pm(2n+1)\frac{\pi}{2} \rightarrow$

$P\left(\pm(2n+1)\frac{\pi}{2}\right) = \sin(\pm(2n+1)\frac{\pi}{2})$

$Q\left(\pm(2n+1)\frac{\pi}{2}\right) = \cos\left(\pm(2n+1)\frac{\pi}{2}\right) \rightarrow$

$$R\left(\pm(2n+1)\frac{\pi}{2}\right) = \frac{\sin(\pm(2n+1)\frac{\pi}{2})}{-\sin(\pm(2n+1)\frac{\pi}{2})}$$

$= -1$

2- $f(z) = \sec(z)$

$$\sec(z) = \frac{1}{\cos(z)}$$

$\cos(z) = 0\ at\ z = \pm(2n+1)\frac{\pi}{2} \rightarrow$

$P\left(\pm(2n+1)\frac{\pi}{2}\right) = 1$

$Q\left(\pm(2n+1)\frac{\pi}{2}\right) = \cos\left(\pm(2n+1)\frac{\pi}{2}\right) \rightarrow$

$$R\left(\pm(2n+1)\frac{\pi}{2}\right) = \frac{1}{-\sin\left(\pm(2n+1)\frac{\pi}{2}\right)}$$

Ex$_{22}$/ show that $\oint \frac{e^{jz}}{\cos(\pi z)} dz = 4sin(\frac{1}{2})$ on the circle $|z| = 1$

Sol:

$\cos(\pi z) = 0$ at $z = \pm\frac{1}{2}(2n+1)$, since $|z| = 1$,then only $z = \pm\frac{1}{2}$ will be considered.

$P(z) = e^{jz}$, $Q\ (z) = \cos(\pi z) \rightarrow \bar{Q}\ (z) = -\pi\ sin(\pi z)$

$$R\left(\pm\frac{1}{2}\right) = \frac{e^{\pm j\frac{1}{2}}}{-\pi\ sin\left(\pm\pi\frac{1}{2}\right)} \rightarrow R\left(\frac{1}{2}\right) = \frac{e^{j\frac{1}{2}}}{-\pi\ sin\left(\pi\frac{1}{2}\right)} \rightarrow R\left(\frac{1}{2}\right) = \frac{e^{j\frac{1}{2}}}{-\pi}$$

In addition $R\left(-\frac{1}{2}\right) = \frac{e^{-j\frac{1}{2}}}{-\pi\ sin\left(-\pi\frac{1}{2}\right)} \rightarrow R\left(\frac{1}{2}\right) = \frac{e^{-j\frac{1}{2}}}{\pi}$

$\therefore R = \frac{e^{j\frac{1}{2}}}{-\pi} + \frac{e^{-j\frac{1}{2}}}{\pi}$

$= \frac{-1}{\pi}(e^{j\frac{1}{2}} - e^{-j\frac{1}{2}})$

$\oint \frac{e^{jz}}{\cos(\pi z)} dz = 2\pi j * R \rightarrow \oint \frac{e^{jz}}{\cos(\pi z)} dz = 2\pi j \frac{-1}{\pi}\left(\frac{e^{j\frac{1}{2}} - e^{-j\frac{1}{2}}}{2j}\right) * 2j$

$\therefore \oint \frac{e^{jz}}{\cos(\pi z)} dz = 4sin(\frac{1}{2})$

❖ Inverse Evaluation of Z-T using Residue Principle

To find $[Z^{-1}[f(z)]]$, the residue theorem can be used as follow

1- Write the equation $f(n) = \frac{1}{2\pi j} \oint f(z) z^{n-1} dz$ ....(26)

2- Find the residue of poles

3- Compute the integral of equation (26)

Ex$_{23}$/ find $f(n)$ for $[f(z) = \frac{z}{z-b}]$ using principle of residue

Sol:

$f(n) = \frac{1}{2\pi j} \oint \frac{z}{z-b} z^{n-1} dz$

$R(b) = \lim_{z\to b}(z-b)\frac{z}{z-b} z^{n-1}$

$= b^n$

I = $\oint \frac{z}{z-b} z^{n-1} dz$

$= 2\pi j(b^n)$

$\therefore f(n) = \frac{1}{2\pi j} * 2\pi j(b^n) \rightarrow f(n) = b^n$

Ex24/ evaluate $f(n)$ for $[\frac{\cos(z)}{(z^2-a^2)e^{-z}}]$using principle of residue

Sol:

$$f(n) = \frac{1}{2\pi j}\oint \frac{\cos(z)}{(z^2-a^2)e^{-z}} z^{n-1}\, dz$$

Since $z^2 - a^2 = (z-a)(z+a)$

$$R(a) = \lim_{z\to a}(z-a)\frac{\cos(z)}{(z-a)(z+a)e^{-z}} z^{n-1}$$

$$= \lim_{z\to a} \frac{\cos(z)}{(z+a)e^{-z}} z^{n-1}$$

$$= \frac{\cos(a)}{(2a)e^{-a}} a^{n-1}$$

In the same way $R(-a)$ can be found as

$$R(-a) = \lim_{z\to -a}(z+a)\frac{\cos(z)}{(z-a)(z+a)e^{-z}} z^{n-1}$$

$$= \lim_{z\to -a} \frac{\cos(z)}{(z-a)e^{-z}} z^{n-1}$$

$$= \frac{\cos(-a)}{(-2a)e^{a}} (-a)^{n-1}$$

$$= \frac{-(-1)^{n-1}\cos(a)}{(2a)e^{a}} (a)^{n-1}$$

$$\therefore R = R(a) + R(-a)$$

$$= \frac{\cos(a)}{(2a)e^{-a}} a^{n-1} - \frac{(-1)^{n-1}\cos(a)}{(2a)e^{a}} (a)^{n-1}$$

$$= \frac{\cos(a)}{(2a)} (a)^{n-1} \left[\frac{1}{e^{-a}} - \frac{(-1)^{n-1}}{e^{a}}\right]$$

$$I = \oint \frac{\cos(z)}{(z^2-a^2)e^{-z}} z^{n-1}\, dz$$

$$= 2\pi j[\frac{\cos(a)}{(2a)} (a)^{n-1} [\frac{1}{e^{-a}} - \frac{(-1)^{n-1}}{e^{a}}]$$

$$\therefore f(n) = \frac{1}{2\pi j} * 2\pi j \frac{\cos(a)}{(2a)} (a)^{n-1} \left[\frac{1}{e^{-a}} - \frac{(-1)^{n-1}}{e^{a}}\right]$$

$$f(n) = \frac{\cos(a)}{(2a)} (a)^{n-1} \left[\frac{1}{e^{-a}} - \frac{(-1)^{n-1}}{e^{a}}\right]$$

HW$_7$: if $f(z) = \left[\frac{z+2}{(z^2-za^2+2)}\right]$ find $f(n)$ using residue theorem

HW$_7$: if $f(z) = \left[\frac{\tan(z)}{\sec(z)}\right]$ find $f(n)$ using residue theorem