❖ Numerical Solution of Non-Linear Systems

Nonlinear algebraic equations are defined as those which contain powers of variables and/or transcendental functions. Such equations arise frequently in engineering, especially when one is dealing with optimization, differential equations, the Eigen problems.

❖ Newton's First Method

$$\tan(\alpha) = \bar{f}(x_o) = \frac{f(x_o)}{x_o - x_1} \rightarrow x_1 = x_o - \frac{f(x_o)}{\bar{f}(x_o)}$$

Or in general

$$x_{n+1} = x_n - \frac{f(x_n)}{\bar{f}(x_n)} \dots\dots\dots (1)$$

Ex$_1$/ solve the following function using Newton's 1st. Method

$$f(x) = 2x^3 - x^2 - 3x - 6$$

Sol:

Since $f(x) = 2x^3 - x^2 - 3x - 6 \rightarrow \bar{f}(x) = 6x^2 - 2x - 3$

Let $x_0 = 1$ for $N = 0$

For $N = 1$ then

$$x_1 = 1 - \frac{(2*1 - 1 - 3 - 6)}{(6*1 - 2*1 - 3)} \rightarrow x_1 = 9$$

$$x_2 = 9 - \frac{(2*9^3 - 1*9^2 - 3*9 - 6)}{(6*9^2 - 2*9 - 3)} \rightarrow x_2 = 6.109677419$$

$$x_3 = 6.109677419 - \frac{(2*6.109677419^3 - 1*6.109677419^2 - 3*6.109677419 - 6)}{(6*6.109677419^2 - 2*6.109677419 - 3)}$$

$$\rightarrow x_3 = 4.220000668$$

In the same way, other values of next iterations can be found to give the following table

N	0	1	2	3	4	5
x	1	9	4.220000668	3.026896761	2.346942781	2.058809301

N	6	7	5	6	7
x	2.0020087872	1.999998335	2.058809301	2.0020087872	1.999998335

It can be found that the final result is approximately equal to (2) which is one of the roots of equation.

Ex$_2$/ solve the following equation using Newton's 1st. Method

$x^2 + 2x + 2 = 0$

Sol:

The exact solution of this equation can be found by the quadratic formula

$x = \frac{-2\pm\sqrt{4-8}}{2} \rightarrow x_{12} = -1 \pm j1$

$f(x) = x^2 + 2x + 2 \rightarrow \bar{f}(x) = 2x + 2$

Start with $x_0 = j$, then

$x_1 = j - \frac{(j^2+2j+2)}{(2j+2)} \rightarrow x_1 = \frac{-3}{4} + j\,\frac{3}{4}$

$x_1 = (\frac{-3}{4} + j\,\frac{3}{4}) - \frac{((\frac{-3}{4}+j\frac{3}{4})^2+2(\frac{-3}{4}+j\frac{3}{4})+2)}{(2(\frac{-3}{4}+j\frac{3}{4})+2)} \rightarrow x_1 = -1.075 + j0.975$

$x_2 = -1.075 + j0.975 - \frac{((-1.075+j0.975)^2+2(-1.075+j0.975)+2)}{(2(-1.075+j0.975)+2)}$

$\rightarrow x_1 = -1.075 + j0.975$

❖ Newton's second Method

For a given which varies continuously over a region where a root exist, a Taylor series can be written with respect to an initial $x_k$value, the value of a function at a new point x_{k+1} is therefore given as

$$f(x_{k+1}) = f(x_k) + \bar{f}(x_k)h + \frac{\bar{\bar{f}}(x_k)h^2}{2!} + \cdots \text{..........} (2)$$

Where $h = (x_{k+1} - x_k)$

It is assumed that the initial value x_k is closed enough to be exact root in equation. Consequently, if equation(2) is converge to a solution, then $f(x_{k+1}) = 0$

$\rightarrow$

$$f(x_k) + \bar{f}(x_k)h + \frac{\bar{f}(x_k)h^2}{2!} = 0 \text{..........} (3)$$

Thus

$$x_{k+1} = x_k - \frac{f(x_k)}{\bar{f}(x_k)} \rightarrow$$

$$h = (x_k - \frac{f(x_k)}{\bar{f}(x_k)} - x_k) \rightarrow h = -\frac{f(x_k)}{\bar{f}(x_k)} \text{..........} (4)$$

divide equation (3) on $(h\, f(x_k)) \rightarrow$

$$\frac{1}{h} = -\frac{\bar{f}(x_k)}{f(x_k)} - \frac{\bar{\bar{f}}(x_k)h}{2f(x_k)} \text{..........} (5)$$

Substituting eq. (4) in RHS of eq. (5)

$$\frac{1}{h} = -\frac{\bar{f}(x_k)}{f(x_k)} - \frac{\bar{\bar{f}}(x_k)}{2f(x_k)}\left(-\frac{f(x_k)}{\bar{f}(x_k)}\right) \text{..........} (5)$$

$\rightarrow$

$$\frac{1}{h} = \frac{\bar{\bar{f}}(x_k)}{2\bar{f}(x_k)} - \frac{\bar{f}(x_k)}{f(x_k)}$$

$$h = \left(\frac{\bar{\bar{f}}(x_k)}{2\bar{f}(x_k)} - \frac{\bar{f}(x_k)}{f(x_k)}\right)^{-1}, \text{ since } h = (x_{k+1} - x_k) \rightarrow$$

$$x_{k+1} = x_k + \left(\frac{\bar{\bar{f}}(x_k)}{2\bar{f}(x_k)} - \frac{\bar{f}(x_k)}{f(x_k)}\right)^{-1}$$

Ex$_3$/ find the root of $\tan(x) - x = 0$ using $x_1 = 5$ by Newton's 2st. Method

Sol:

$f(x) = \tan(x) - x$

$\bar{f}(x) = \sec^2(x) - 1$

$= \frac{1}{\cos^2(x)} - 1 = \tan^2(x)$

$\bar{\bar{f}}(x) = 2\tan(x)\sec^2(x) \rightarrow 2\tan(x)\sec^2(x) = 2\,\frac{\tan(x)}{\cos^2(x)}$

$x_1 = 5$

$f(5) = \tan(x) - x \rightarrow f(5) = -8.3805$

$\bar{f}(5) = \tan^2(x) \rightarrow \bar{f}(5) = 11.4279$

$\bar{\bar{f}}(5) = 2\,\frac{\tan(x)}{\cos^2(x)} \rightarrow \bar{\bar{f}}(5) = -84.0252$

$x_2 = 5 + (\frac{11.4279}{-8.3805} + \frac{1}{2}\frac{-84.0252}{11.4279})^{-1} \rightarrow x_2 = 4.5676$

$f(x_2) = 2.2908$

$\bar{f}(x_2) = 47.0382$

$\bar{\bar{f}}(x_2) = 658.89$

And so on to achieve gre3ater accuracy.

❖ ***Newton's second Method for system of equations***

For the following equation

$x^3 + Ax^2 + Bx + C = 0$

Which can be expressed as a system as follow

$(x - x_1)(x - x_2)(x - x_3) = x^3 + Ax^2 + Bx + C$

$x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3 = x^3 + Ax^2 + Bx + C$

$\rightarrow -(x_1 + x_2 + x_3) = A$

$(x_1x_2 + x_1x_3 + x_2x_3) = B$

$-x_1x_2x_3 = C$

$\rightarrow$

$f_1(x_1, x_2, x_3) = A + x_1 + x_2 + x_3$

$f_2(x_1, x_2, x_3) = B - x_1x_2 - x_1x_3 - x_2x_3$

$f_3(x_1, x_2, x_3) = C + x_1x_2x_3$

Let $[J] = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_1} & \frac{d2}{dx_1} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{bmatrix}$

The matrix $[J]$ is called the Jacobean, the general iteration form is

$[x_{k+1}] = [x_k] - [J]^{-1}[f(x)]$

Ex$_4$/ approximate the roots to the nonlinear algebraic equation into the equivalent system of nonlinear algebraic equations.

$f(x) = x^3 + 2x^2 + x - 2$

Take $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix}$

Sol:

$f_1(x_1, x_2, x_3) = 2 + x_1 + x_2 + x_3$

$f_2(x_1, x_2, x_3) = 1 - x_1x_2 - x_1x_3 - x_2x_3$

$f_3(x_1, x_2, x_3) = -2 + x_1x_2x_3$

$\rightarrow$

$[J] = \begin{bmatrix} 1 & 1 & 1 \\ -x_2 - x_3 & -x_1 - x_3 & -x_1 - x_2 \\ x_2x_3 & x_1x_3 & x_1x_2 \end{bmatrix}$

For $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_1 = \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix}$

$[J]_1 = \begin{bmatrix} 1 & 1 & 1 \\ -2.5 & -2 & -1.5 \\ 1.5 & 0.75 & 0.5 \end{bmatrix} \rightarrow [J]^{-1}{}_1 = \begin{bmatrix} 0.5 & 1 & 2 \\ -4 & -4 & -4 \\ 4.5 & 3 & 2 \end{bmatrix}$

For $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_1 = \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix}$, find $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}_1 = \begin{bmatrix} 5 \\ -1.57 \\ -1.25 \end{bmatrix}$

$\rightarrow$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_2 = \begin{bmatrix} 0.5 \\ 1 \\ 1.5 \end{bmatrix}_1 - \begin{bmatrix} 0.5 & 1 & 2 \\ -4 & -4 & -4 \\ 4.5 & 3 & 2 \end{bmatrix}_1 \begin{bmatrix} 5 \\ -1.57 \\ -1.25 \end{bmatrix}$

$= \begin{bmatrix} 2.07 \\ 9.72 \\ -13.79 \end{bmatrix}_2$, for $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_2 \rightarrow \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 143.5 \\ -279.5 \end{bmatrix}$

$[J]_2 = \begin{bmatrix} 1 & 1 & 1 \\ 4.07 & 11.72 & 11.79 \\ 134.04 & 28.5 & 20.12 \end{bmatrix} \rightarrow$

$[J]^{-1}{}_2 = \begin{bmatrix} 1.8 & -0.15 & -1.23 \\ -26.41 & 2 & 0.14 \\ 25.6 & -1.8 & -0.13 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_3 = \begin{bmatrix} 2.07 \\ 9.72 \\ -13.79 \end{bmatrix}_2 - \begin{bmatrix} 01.81 & -0.15 & -1.4 \\ -30.86 & 2.35 & 0.16 \\ 30.06 & -2.2 & -0.16 \end{bmatrix}_2 \begin{bmatrix} 0 \\ 143.5 \\ -279.5 \end{bmatrix}$

$= \begin{bmatrix} -320 \\ -238 \\ 208.2 \end{bmatrix}_3$

And so on to obtain more accuracy.

❖ Interpolation

Anyone who has had occasion to consult tables of mathematical functions is familiar with the method of linear interpolation and probably has encountered situations in which this method of "reading between the lines of the table" has appeared to be unreliable.

❖ Linear Interpolation

The assumption that a function $f(x)$ is approximately linear, in a certain range, is equivalent to the assumption that the ratio [$\frac{f(x_1)-f(x_0)}{x_1-x_0}$]

In this case the interpolation is an interpolation by airtight line through $(x_0, f(x_0))$ &$(x_1, f(x_1))$

$f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0}$

$f_1(x) = f(x_0) + (x - x_0)\, f(x_0, x_1)$

Ex$_5$/ find $\sqrt{5}$ from ($\sqrt{4} = 3$) and ($\sqrt{9} = 2$) using linear interpolation.

Sol:

$x_0 = 4 \rightarrow f(x_0) = 2$ & $x_1 = 9 \rightarrow f(x_1) = 3$

since $f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} \rightarrow f(x_0, x_1) = \frac{3-2}{9-4}$

= 0.2

$f_1(x) = 2 + (x - 4) * 0.2 \rightarrow f_1(5) = 2.2$

The exact value (2.24) , the error = 0.036

Ex$_6$/ repeat example (5) from ($\sqrt{4.5} = 2.121$) and ($\sqrt{5.5} = 2.345$)

Sol:

$x_0 = 4.5 \rightarrow f(x_0) = 2.121$ & $x_1 = 5.5 \rightarrow f(x_1) = 2.345$

since $f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} \rightarrow f(x_0, x_1) = \frac{2.345-2.121}{5.5-4.5}$

= 0.224

$f_1(x) = 2.121 + (x - 4.5) * 0.224 \rightarrow f_1(5) = 2.2332$

The exact value (2.24) , the error = 0.00286

from the previous two examples, it is found that the error between the exact value and the interpolation value reduces when the two values (x_0, x_1) are near to the wonted value.

Ex7/ find $e^{-1.5}$ from ($e^{-1} = 0.3679$) and ($e^{-2} = 0.135335$) using linear interpolation.

Sol:

$x_0 = -1 \rightarrow f(x_0) = 0.3679$ & $x_1 = -2 \rightarrow f(x_1) = 0.135335$

since $f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} \rightarrow f(x_0, x_1) = \frac{0.135335 - 0.3679}{-2+1}$

= 0.23257

$f_1(x) = 0.3679 + (-2+1) * 0.23257 \rightarrow f_1(x) = 0.13533$

The exact value (0.2313) , the error = 0.0696

❖ ***Quadratic Interpolation***

In this case

$f_2(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$

$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$

Ex8/ find $\sqrt[3]{7}$ from [$\sqrt[3]{6} = 1.82, \sqrt[3]{8} = 2, \sqrt[3]{9} = 2.01$] using quadratic interpolation.

$x_0 = 6 \quad f(x_0) = 1.82$

$f(x_0, x_1) = 0.09$

$x_1 = 8 \quad f(x_1) = 2 \qquad f(x_0, x_1, x_2) = -0.0267$

$f(x_1, x_2) = 0.01$

$x_2 = 9 \quad f(x_2) = 2.01$,Since

$f_2(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$

$\rightarrow$

$f_2(7) = 1.82 + (7 - 6) * 0.09 + (7 - 6)(7 - 8) * -0.0267$

$\therefore f_2(7) = 1.9367$, since the exact vale = 1.91229

then the error = 0.02377

❖ ***Newton Divided difference interpolation***

To find the function of (x) with order (n), the following equation is used

$$f_n(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \dots . +(x - x_0) \dots . (x - x_{n-1})f(x_0, x_1, \dots \dots . . x_n)$$

Ex9/ if [$e^{-0.5} = 0.606, e^{-0.75} = 0.472, e^{-1} = 0.367, e^{-1.5} = 0.223$, $e^{-1.75} = 0.174, e^{-2} = 0.14$, find $e^{-1.25}$

Sol:

$x_0 = -0.5 \rightarrow f(x_0) = 0.606$

$f(x_0, x_1) = 0.536$

$x_1 = -0.75 \rightarrow f(x_1) = 0.472$ $\quad f(x_0, x_1, x_2) = 0.232$

$f(x_1, x_2) = 0.42$ $\quad f(x_0, x_1, x_2, x_3) = 0.056$

$x_2 = -1 \rightarrow f(x_2) = 0.367$ $\quad f(x_1, x_2, x_3) = 0.176$ $\quad f(x_0, x_1, x_2, x_3, x_4) = 0.0024$

$f(x_2, x_3) = 0.288$ $\quad f(x_1, x_2, x_3, x_4) = 0.053$ $\quad f(x_0, x_1, x_2, x_3, x_4, x_5) = -0.0251$

$x_3 = -1.5 \rightarrow f(x_3) = 0.223$ $\quad f(x_2, x_3, x_4) = 0.123$ $\quad f(x_1, x_2, x_3, x_4, x_5) = 0.04$

$f(x_3, x_4) = 196$ $\quad f(x_2, x_3, x_4, x_5) = 0.003$

$x_4 = -1.75 \rightarrow f(x_4) = 0.174$ $\quad f(x_3, x_4, x_5) = 0.12$

$f(x_4, x_5) = 0.136$

$x_5 = -2 \rightarrow$ $\quad f(x_5) = 0.14$

$$f_5(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + +(x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)f(x_0, x_1, x_2, x_3, x_4) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)f\left(x_0, x_1, x_2, x_3, x_4, x_5\right)$$

$e^{-1.25}$= $0.606+(-1.25 + 0.5)*0.536+(-1.25 + 0.5)(-1.25 + 0.75) * 0.232+(-1.25 + 0.5)(-1.25 + 0.75)(-1.25 + 1)*0.056+(-1.25 + 0.5)(-1.25 + 0.75)(-1.25 + 1)(-1.25 + 1.5) * 0.0024+(-1.25 + 0.5)(-1.25 + 0.75)(-1.25 + 1)(-1.25 + 1.5)(-1.25 + 1.75) * -0.0251 \rightarrow$ $e^{-1.25}$= 0.286269

While the exact value =0.286505, this means that the error is (0.000236)

Ex10/ find $ln\,(1.5)$, if $ln(0.5) = -0.693$, $ln(1) = 0$, $ln(2) = 0.693$, $ln(2.5) = 0.9163$.

$x_0 = 0.5 \rightarrow f(x_0) = -0.693$

$f(x_0, x_1) = 1.386$

$x_1 = 1 \rightarrow f(x_1) = 0$ $\quad f(x_0, x_1, x_2) = -\,0.462$

$f(x_1, x_2) = 0.693$ $\quad f(x_0, x_1, x_2, x_3) = \ 0.1486$

$x_2 = 2 \rightarrow f(x_2) = 0.693$ $\quad f(x_1, x_2, x_3) = \ -0.1646$

$f(x_2, x_3) = 0.446$

$x_3 = 2.5 \rightarrow f(x_3) = 0.916$

$f_3(x) = \ f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) +$
$+(x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$

$\rightarrow$

$ln\,(1.5) = \ -0.693 \quad + \quad (1.5 - 0.5) * 1.386 + (1.5 - .5)(1.5 - 1) *$
$-\,0.462 + \ (1.5 - .5)(1.5 - 1)(1.5 - 2) * -0.184$
$= 0.4248$

The exact value $= 0.405$, this means that the error is (0.103)

❖ ***Forward interpolation***

Suppose there are a set of n data points relating a dependent variable $f(x)$ to an independent variable $x$as fpllows:

x_i	x_0	x_1	…….	…….	x_n
$f(x_i)$	$f(x_0)$	$f(x_1)$	…….	…….	$f(x_n)$

Generally, the base points ($x_0, \ldots\ldots, x_n$) are arbitrary, lets assume that the interval between two adjacent points is fixed, then

$h = x_1 - x_0 = \ x_2 - x_1 = \cdots = x_n - x_{n-1}$ or in general $h = x_{i+1} - x_i$

for the second function equation

$f(x) \approx \ a_0 + a_1 x + \ a_2 x^2$ ……… (6)

Let the base point ($x_0 = 0$) which is related to other points forward of it. Substituting the three data points into eq. (6) yields three linear algebraic equations as follows:

$f_0 = a_0$

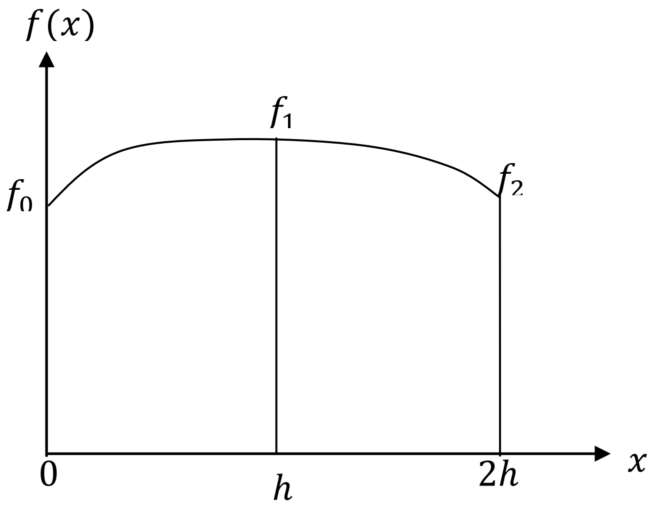
$f_1 = a_0 + ha_1 + h^2 a_2$

$f_2 = a_0 + 2ha_1 + 4h^2 a_2$

Note that $f_0 = f(x_0), f_1 = f(x_1), f_2 = f(x_2) \rightarrow$

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix}$$

The forward interpolating function of the second order is given in the following figure



By taking the invention of the coefficients matrix then

$$\begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \text{ from which}$$

$a_0 = f_0$

$a_1 = \frac{1}{2h}(-3f_0 + 4f_1 - f_2)$ and $a_2 = \frac{1}{2h^2}(f_0 - 2f_1 + f_2)$

Substituting in equation (6) to gate

$f(x) \approx f_0 + \frac{1}{2h}(-3f_0 + 4f_1 - f_2)x + \frac{1}{2h^2}(f_0 - 2f_1 + f_2)x^2$ ....(7)

Equation(7) is called forward interpolation formula.

At

$x = 0 \rightarrow f(0) = f_0$

$x = h \rightarrow f(h) = f_1$

$x = 2h \rightarrow f(2h) = f_2$

Ex11/ for the following data, approximate the functional value at $x = 2$, using forward interpolation.

x	1	3	5
$f(x)$	2	4	8

Sol:

$x_0 = 0$, $x_1 = h$, $x_2 = 2h$, where $h = 2$, subtracting (1) from each base point to give

i	Given data		Transformed data	
	x_i	$f(x_i) = f_i$	$(x_i)_{tr}$	$f(x_i) = f_i$
0	1	2	0	2
1	3	4	2	4
2	5	8	4	8
	2	?	1	?

Since $f(x) = f_0 + \frac{1}{2h}(-3f_0 + 4f_1 - f_2)x + \frac{1}{2h^2}(f_0 - 2f_1 + f_2)x^2$

$\rightarrow f(x) = 2 + \frac{1}{4}(-3 * 2 + 4 * 4 - 8)x + \frac{1}{8}(2 - 2 * 4 + 8)x^2$

$\therefore f(x) = 2 + 0.5x_{tr} + 0.25\, x^2{}_{tr}$

Therefore, at ($x_{tr} = 1$) it can be found that

$f(1) = 2.75$

And since $h = 2$, then substituting $(x-2)$

in $f(x) = 2 + 0.5x + \ 0.25\,x^2$ gives that

$f(x) = 2 - 0.5x + \ 0.25\,x^2$

- ***Backward interpolation***

When the three base points are $(0, -h, -2h)$ with corresponding functional values (f_0, f_{-1}, f_{-2}), substituting in eq.(6) gives that

$f_0 = a_0$, $f_1 = \ a_0 - ha_1 + h^2\,a_2$, and $f_2 = \ a_0 - 2ha_1 + 4h^2\,a_2$

$$\begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} f_0 \\ f_{-1} \\ f_{-2} \end{bmatrix} \rightarrow$$

$$f(x) = \ f_0 \ + \frac{1}{2h}(f_{-2} - 4f_{-1} + 3f_0)x + \ \frac{1}{2h^2}(f_{-2} - 2f_{-1} + f_0)x^2 \text{.....(8)}$$

Ex12/ for the following data, approximate the functional value at $x \ = 5$, using forward interpolation.

x	2	4	6
$f(x)$	8	2	8

Sol:

$x_0 = 0$, $x_{-1} = -h, x_{-2} = -2h$, where $h = 2$, subtracting (6) from each base point to give

i	Given data		Transformed data	
	x_i	$f(x_i)= f_i$	$(x_i)_{tr}$	$f(x_i)= f_i$
-2	2	8	-4	2
-1	4	2	-2	4
0	6	8	0	8
	5	?	-1	?

$\rightarrow$

$f(x) = 2 + 6x_{tr} + \ 1.5\,x^2{}_{tr}$

Therefore, at ($x_{tr} = -1$) it can be found that

$f(-1) = 3.5$

❖ ***Central interpolation***

In this type, the base points are $(-h, 0, h)$and their corresponding functional values are (f_{-1}, f_0, f_1) this type of interpolating is used with second order polynomials because for odd order polynomials, it is impossible to have a central base point, substituting in eq.(6) gives

$f_{-1} = a_0 - ha_1 + h^2\, a_2$

$f_0 = a_0$

$f_1 = a_0 + ha_1 + h^2\, a_2$

$$\begin{bmatrix} a_0 \\ a_1 h \\ a_2 h^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} f_{-1} \\ f_0 \\ f_1 \end{bmatrix}$$

Substituting (a_0, a_1, a_2) in eq. (6) gives

$$f(x) = f_0 + \frac{1}{2h}(-f_{-1} + f_1)x + \frac{1}{2h^2}(f_{-1} - 2f_0 + f_1)x^2 \text{.......(8)}$$

$$f(x) = [1 \quad x \quad x^2]\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \text{.......(9)}$$

Therefor the forward interpolating function can be easly determined by noting that

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ \frac{-3}{h} & \frac{4}{h} & \frac{-1}{h} \\ \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

substituting in eq.(9)

$$f(x) = \frac{1}{2}[1 \quad x \quad x^2]\begin{bmatrix} 2 & 0 & 0 \\ \frac{-3}{h} & \frac{4}{h} & \frac{-1}{h} \\ \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

$$=\frac{1}{2}\left(2-\frac{3x}{h}+\frac{x^2}{h^2}\right)f_0+\left(2\frac{x}{h}-\frac{x^2}{h^2}\right)f_1+\frac{1}{2}\left(\frac{x}{h}+\frac{x^2}{h^2}\right)f_2$$

The interpolating formula may be expressed as follows:

$f(x) = N_0 f_0 + N_1 f_1 + N_2 f_2$

$= \sum_{i=0}^{2} N_i f_i$

$N_0=\frac{1}{2}\left(2-\frac{3x}{h}+\frac{x^2}{h^2}\right)=N_0(x)$

$N_1=\left(2\frac{x}{h}-\frac{x^2}{h^2}\right)=N_1(x)$

$N_2=\frac{1}{2}\left(-\frac{x}{h}+\frac{x^2}{h^2}\right)= N_2(x)$

These functions are called shape functions, which has the following properties:

At $x=0 \rightarrow N_0=1, N_1=0, N_2=0$

At $x=h \rightarrow N_0=0, N_1=1, N_2=0$

At $x=2h \rightarrow N_0=0, N_1=0, N_2=1$

For any (x)

$N_0+N_1+N_2=1$

In general, for any polynomial of nth order, the corresponding interpolating function is given as:

$f(x) = N_0 f_0 + N_1 f_1 + N_2 f_2$

$f(x) = N_0 f_0 + N_1 f_1 + \ldots\ldots + N_n f_n = \sum_{i=0}^{n} N_i f_i$

$\rightarrow \sum_{i=1}^{n} N_i = 1$

$N_i(x_j) = 0 \quad for\ i \neq j$

$\qquad\quad =1 \quad for\ i = j$

Ex12/ for the following data, approximate the functional value at $x = 6$, using forward interpolation.

x	3	4	5
$f(x)$	4	2	4

Sol:

$x_0 = 0$, $x_{-1} = -h, x_1 = h,$ where $h = 1,$ subtracting (4) from each base point to give

i	Given data		Transformed data	
	x_i	$f(x_i) = f_i$	$(x_i)_{tr}$	$f(x_i) = f_i$
-1	3	4	-1	4
0	4	2	0	2
1	5	4	1	4
	6	?	2	?

Since $f(x) = f_0 + \frac{1}{2h}(-f_{-1} + f_1)x + \frac{1}{2h^2}(f_{-1} - 2f_0 + f_1)x^2$, then

$f(x) = 2 + 0.5x^2{}_{tr}$

For $x_{tr} = 2 \rightarrow f(2) = 3$

Using shape functions to obtain the following result:

$f(x) = 4N_0 + 2N_1 + 4N_2$, for $x_{tr} = 3$

$N_0 = \frac{1}{2}\left(2 - \frac{3*3}{1} + \frac{(3)^2}{(1)^2}\right)$

$N_1 = \left(2\frac{3}{1} - \frac{(3)^2}{(1)^2}\right)$

$N_2 = \frac{1}{2}\left(-\frac{3}{1} + \frac{(3)^2}{(1)^2}\right)$

$N_0(3) = 1$

$N_1(3) = -3$

$N_2(3) = 3$

$\rightarrow f(3) = 10$