

Quine-McClusky Minimization Procedure

This is basically a tabular method of minimization and as much it is suitable for computer applications. The procedure for optimization as follows:

- Step 1:** Describe individual minterms of the given expression by their equivalent binary numbers.
- Step 2:** Form a table by grouping numbers with equivalent number of 1's in them, i.e. first numbers with no 1's, then numbers with one 1, and then numbers with two 1's, ... etc.
- Step 3:** Compare each number in the top group with each minterm in the next lower group. If the two numbers are the same in every position but one, place a check sign (✓) to the right of both numbers to show that they have been paired and covered. Then enter the newly formed number in the next column (a new table). The new number is the old numbers but where the literal differ, an "x" is placed in the position of that literal.
- Step 4:** Using (3) above, form a second table and repeat the process again until no further pairing is possible. (On second repeat, compare numbers to numbers in the next group that have the same "x" position.
- Step 5:** Terms which were not covered are the prime implicants and are ORed and ANDed together to form final function.

Note: The procedure above gives you the prime implicant but not **essential** prime implicant.

Example 1

Minimize the function given below by Quine-McClusky method.

f(A,B,C,D)=	$\bar{A}\bar{B}\bar{C}\bar{D}+$	$\bar{A}BCD+$	$\bar{A}BC\bar{D}+$	$\bar{A}\bar{B}CD+$	$A\bar{B}\bar{C}\bar{D}+$	$AB\bar{C}\bar{D}+$	$ABC\bar{D}+$	$ABCD+$	$\bar{A}BCD$
Binary	0000	0101	0110	1001	1010	1101	1110	1111	0111
minterm	0	5	6	9	10	13	14	15	7
No of 1's	0	2	2	2	2	3	3	4	3
group	1	2	2	2	2	3	3	4	3

This can be written as a sum of minterms as follows:

$$f(A, B, C, D) = \sum m(0, 5, 6, 7, 9, 10, 13, 14, 15)$$

- Step 1:** Form a table of functions of minterms according to the number of 1's in each minterm as shown in Table E1.a

minterm	A	B	C	D		
* 0	0	0	0	0	} All numbers with no 1's in each minterm (a)	
5	0	1	0	1		✓✓...
6	0	1	1	0	✓✓...	} All numbers with two 1's in each minterm
9	1	0	0	1	✓✓...	
10	1	0	1	0	✓✓...	
7	0	1	1	1	✓✓...	} All numbers with three 1's in each minterm
13	1	1	0	1	✓✓...	
14	1	1	1	0	✓✓...	
15	1	1	1	1	✓	

CD \ AB	00	01	11	10
00	1 ⁰		4	12
01	1	1 ⁵	1 ¹³	1 ⁹
11	3	1 ⁷	1 ¹⁵	11
10	2	1 ⁶	1 ¹⁴	1 ¹⁰

Table E1.a

Step 2: Start pairing off each element of first group with the next, however since m_0 has no 1's, it and the next group of numbers with one 1's are missing, therefore they cannot be paired off. Start by pairing elements of m_5 with m_7 , m_{13} , m_{14} , and m_6 with m_7 , m_{13} , m_{14} , and so on... If they pair off, write them in a separate table and ✓ the minterm that pair, i.e. m_5 and m_7 pair off 0101 and 0111 to produce 01x1, so in the next table E1.b under "minterm paired" we enter "5, 7" and under "ABCD" we enter "01x1" and place a ✓ sign in front of 5 and 7 in Table E1.a

Note: Each minterm in a group must be compared with every minterm in the other group even if either or both of them have already been checked ✓.

minterms paired	A	B	C	D	
5, 7	0	1	X	1	✓
5, 13	X	1	0	1	✓✓
6, 7	0	1	1	X	✓✓
6, 14	X	1	1	0	✓✓
9, 13	1	X	0	1 (b)
10, 14	1	X	1	0 (c)
7, 15	X	1	1	1	✓✓
13, 15	1	1	X	1	✓✓
14, 15	1	1	1	X	✓✓

Table E1.b

Paired minterms from E1.b	A	B	C	D	
5,7 – 13,15	x	1	x	1	... (d)
6,7 – 14,15	x	1	1	x	... (e)

Table E1.c

Step 3: Now repeat the same procedure by pairing each element of a group with the elements of the next group for elements that have "x" in the same position. For example, "5,7" matches "13,15" to produce x1x1. These elements are placed in table E1.c as shown, and the above elements in Table E1.b are ✓ checked. (The elements that produce the same ABCD pattern are eliminated.) Since 9,13 and 10,14 in Table E1.b do not pair off, they are prime implicants and with m_0 , from E1.a, and (d) and (e) from E1.c are unpaired individuals. Therefore, it is possible to write the minimized SOP as $a+b+c+d+r$ or

$$f(A, B, C, D) = \overline{A}\overline{B}\overline{C}\overline{D} + \overline{A}CD + A\overline{C}\overline{D} + BD + BC$$

Note: Check this result for Example 1 by Karnaugh map approach.

Two-square implicants:

CD \ AB	00	01	11	10
00	1 0	4	12	8
01	1	1 5	1 13	1 9
11	3	1 7	1 15	11
10	2	1 6	1 14	1 10

Table E1.b represents all possible two-square implicants and the literals that they eliminate, i.e. 9 (1001_b) combined with 13 (1101_b) produces 1x01. As a result, literal "B" is eliminated. Corresponding product is $\overline{A}CD$. Since the only way of making an implicant that contains m₉ is to combine it with m₁₃, the implicant 9-13 is a prime one. The same rule applies to m₁₀.

Four-square implicants:

CD \ AB	00	01	11	10
00	1 0	4	12	8
01	1	1 5	1 13	1 9
11	3	1 7	1 15	11
10	2	1 6	1 14	1 10

Table E1.c represents all possible four-square implicants and the literals that they eliminate, i.e. 5 (0101_b) combined with 7 (0111_b) and 13 (1101_b) and 15 (1111_b) produces x1x1. As a result, literals "A" and "C" are eliminated. Corresponding product is BD.

Quine-McClusky Minimization Procedure (the decimal notation)

- Step 1:** List the minterms grouped according to the number of 1's in their binary representation in the decimal format.
- Step 2:** Compare each minterm with larger minterms in the next group down. If they differ by a power of 2 then they pair-off. Check both minterms and form a second table by the minterms paired and substitute the decimal difference of the corresponding minterms in the bracket, i.e. $m_x, m_y (y-x)$.
- Step 3:** Compare each element of the group in the new table with elements of the next lower group and select numbers that have the same numbers in parenthesis. If the lowest minterm number of the table formed in the lower group is greater than the corresponding number by a power of 2 then they combine; place a ✓ on the right of both elements.
- Step 4:** Form a second table by all four minterms followed by both powers of 2 in parentheses, i.e. the previous value (the difference) and the power of 2 that is greater.
- Step 5:** Select the common literals from each prime implicant by comparison.
- Step 6:** Write the minimal SOP from the prime implicant that are not checked ✓.

Note: Read the above procedure in conjunction with the worked example given below.

Example E1.1

Minimize the function f given below by Quine-McClusky method using decimal notation.

$$f(A, B, C, D) = \overline{A}BCD + \overline{A}BC\overline{D} + \overline{A}B\overline{C}D + \overline{A}B\overline{C}\overline{D} + \overline{A}B\overline{C}D + \overline{A}B\overline{C}\overline{D} + \overline{A}BCD + \overline{A}BC\overline{D} + \overline{A}BCD + \overline{A}BC\overline{D} + ABCD + ABC\overline{D} + ABCD + ABC\overline{D} + ABCD + ABC\overline{D} + ABCD + ABC\overline{D}$$

Solution

Step 1: Organize minterm as follows:

$$f(A, B, C, D) = \sum m(0, 5, 6, 7, 9, 10, 13, 14, 15)$$

Arrange minterms to correspond to their number of 1's as shown in E1.1a

1's	Minterms
* 0	0 ... (a)
2	5 ✓
	6 ✓
	9 ✓
	10 ✓
3	7 ✓
	13 ✓
	14 ✓
4	15 ✓

Table E1.1a

minterm	paired	
5 ^φ , 7 ^φ	(2) [†]	✓
5, 13	(8)	✓
6, 7	(1)	✓
6, 14	(8)	✓
* 9, 13	(4)	... (b)
* 10, 14	(4)	... (c)
7, 15	(8)	✓
13, 15	(2)	✓
14, 15	(1)	✓

Table E1.1b

minterms	paired	
* 5, 7-13, 15 [†]	(2, 8)	... (d)
* 6, 14-7, 15	(1, 8)	... (e)

Table E1.1c

^φ - squares combined (2 squares);

† - number in bracket shows the literal being eliminated, i.e. (2) represents C [A=8, B=4, C=2, D=1];

i- squares combined (4 squares) and numbers in the brackets are the literals eliminated.

Step 2: Compare each element of a group with the element of the next group if the difference is a power of 2 then they pair off, i.e. the first element in group 2 is paired say with the first element in group 3, which is $7-5=2$, which is power of 2. Therefore, pair (5,7) makes the first element of the next table and minterms 5 and 7 get checked ✓. The result is shown in Table E1.1b.

Step 3: Now for the 2^3 -table again compare each element of the group with elements of the lower group that have the same number in parentheses. If the lowest minterm in the lower group was greater by a power of 2 then they combine, i.e. 5,7 and 13,15 are combined because they have (2) in parentheses and 13 is greater than 5 by 8. Then they are paired off and entered in the next table E1.1c with the original (2) and their difference (8) in the parentheses.

Step 4: What we are left with is (a) from Table E1.1a, (b) and (c) from Table E1.1b, and (d) and (e) from Table E1.1c. From Table E1.1c, "d" is 5,7-13,15 (2,8). That means that positions 2^1 and 2^3 are X's. Thus, "d" represents function BD. From the same table, "e" is 6,14-7,15 (1,8), which means positions 2^0 and 2^3 are X's. . Thus, "e" represents function BC. This can also be obtained by writing the elements of minterms and selecting two remaining literals:

{	6	0	1	1	0
	14	1	1	1	0
	7	0	1	1	1
	15	1	1	1	1
		x	1	1	x
			B	C	

Therefore, the minimized SOP is

$$f = a + b + c + d + e = \overline{A}\overline{B}\overline{C}\overline{D} + A\overline{C}\overline{D} + A\overline{C}\overline{D} + BD + BC$$

Note: Compare this with the method of K-map or standard Quine-McClusky (the first approach).

The above function consists of prime implicants. However, not all of them are necessary essential prime implicants.

Example 1.1.1. Determination of Essential Prime Implicants

For the SOP obtained in Example 1.1, determine the essential prime implicants and see if further reduction is possible.

Solution.

Construct a prime implicants table as shown in Table 1.1.1a, with prime implicants on left and minterms on top:

	Minterms	0	5	6	7	9	10	13	14	15
*₍₂₎	5, 7 – 13, 15		✓		✓			✓		✓
*₍₃₎	6, 14 – 7, 15			✓	✓				✓	✓
*₍₄₎	9, 13					✓		✓		
*₍₅₎	10, 14						✓		✓	
*₍₁₎	0	✓								
		✓ ₍₁₎	✓ ₍₂₎	✓ ₍₃₎		✓ ₍₄₎	✓ ₍₅₎			
	5		✓		✓			✓		✓
	6			✓					✓	
	9					✓				

Table 1.1.1a

In each row, (except the bottom) checks ✓ are placed in the columns corresponding to minterms contained in the prime implicant listing in that row, i.e. the first prime implicant testing contains 5, 7, 13, 15. So, ✓ is placed in the first row in columns 5, 7, 13, 15. Repeat for each prime implicant.

Now inspect the table for columns that contain only one ✓. That means that that prime implicant is the only term that contains that minterm, i.e. for example m_0 must be included in the SOP. This is marked with asterisks (*) in the left column and place ✓ in the bottom row. The same applies to 6, 9, and 10. Therefore, all prime implicants in this example are essential prime implicants. Other empty cells in the bottom row are covered by essential prime implicants. For example, once 5 is selected, then 7, 13, and 15 also can be ✓ from the bottom row, and so on.